KINGS
ENGINEERING COLLEGE
An Autonomous Institution
Affiliated to Anna University, Chennai

Regulation2024

II Year – III Semester

MA242303 Transforms and Random Processes

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REGULATIONS 2024

II YEAR/ III SEMESTER

(COMMON TO BME)

MA 242303 TRANSFORMS AND RANDOM PROCESSES

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OBJECTIVES:

- To make the student acquire sound knowledge of Laplace Transform techniques in solving ordinary differential equations.
- To acquaint the student with Fourier transform techniques used in wide variety of situations.
- To introduce the effective mathematical tools for the solutions of partial differential equations that model several physical processes and to develop Z transform techniques for discrete time systems.
- To introduce the basic concepts of random variables.
- To understand the basic concepts of random processes which are widely used in IT and electronics fields.

UNIT I LAPLACE TRANSFORMS 9 + 3

Existence conditions – Transforms of elementary functions – Basic properties – Change of Scale property and First Shifting theorem – Inverse transforms – Partial fraction method – Convolution theorem based on Inverse Laplace transform — Solution of linear second order ordinary differential equations with constant coefficients.

UNIT II FOURIER TRANSFORMS 9 + 3

Fourier transform pair – Fourier sine and cosine transforms pair – Properties of Fourier Transform – Transforms of simple functions – Convolution theorem based on Inverse Fourier transform – Parseval's identity of Fourier transform.

UNIT III Z - TRANSFORMS AND DIFFERENCE EQUATIONS 9 + 3

Z-transforms - Elementary properties of Z-transforms – Formation of difference equation – Inverse Z-transform using partial fraction – Convolution theorem based on Inverse Z-transform – Solution of difference equations using Z - transform.

UNIT IV RANDOM VARIABLES 9 + 3

Concepts of Probability – The axioms of probability – Random Variables – Discrete and Continuous random variables, Probability mass function and Probability density function – Cumulative Distribution Function - Moments – Moment generating functions – Binomial, Poisson, Uniform and Exponential distributions.

UNIT V RANDOM PROCESSES 9 + 3

Classification of Random processes – Stationary process – Average values of Random processes (Mean and Autocorrelation) - Wide sense stationary process – Markov process - Markov chain, Formation of transition probability matrix, Classification of states of a Markov chain, Long run – limiting case, Markov diagram.

TOTAL: 60 PERIODS

OUTCOMES :

Upon successful completion of the course, students should be able to:

CO1: Solve Laplace transform and inverse transform of simple functions, properties, various related theorems and application to differential equations with constant coefficients.

CO2: Understand the mathematical principles on Fourier transforms and partial differential equations would provide them the ability to formulate and solve some of the physical problems of engineering.

CO3: Use the effective mathematical tools for the solutions of partial differential equations by using Z transform techniques for discrete time systems.

CO4: Understand the fundamental knowledge of the concepts of probability and have knowledge of standard distributions which can describe real life phenomenon.

CO5: Understand the concept of random processes in engineering disciplines.

TEXTBOOKS

1. Grewal B.S., “Higher Engineering Mathematics", 45th Edition, Khanna Publishers, New Delhi, 2024.
2. Johnson, R.A., Miller, I and Freund J., "Miller and Freund’s Probability and Statistics for Engineers", Pearson Education, Asia, 8th Edition, 2015.

REFERENCES :

1. Andrews, L.C and Shivamoggi, B, "Integral Transforms for Engineers" SPIE Press, 1999.
2. Bali. N.P and Manish Goyal, "A Textbook of Engineering Mathematics", 9th Edition, Laxmi Publications Pvt. Ltd, 2014.
3. Erwin Kreyszig, "Advanced Engineering Mathematics ", 10th Edition, John Wiley, India, 2016.
4. Miller. S.L. and Childers. D.G., ―Probability and Random Processes with Applications to Signal Processing and Communications ", Academic Press, 2004.
5. Papoulis, A. and Unnikrishnapillai, S., "Probability, Random Variables and Stochastic Processes", McGraw Hill Education India, 4th Edition, New Delhi, 2010.

CO’s-PO’s & PSO’s MAPPING

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12	PSO1	PSO2	PSO3
CO1	3	3	-	-	-	-	-	-	-	-	-	-	-	-	-
CO2	3	3	-	-	-	-	-	-	-	-	-	-	-	-	-
CO3	3	3	-	-	-	-	-	-	-	-	-	-	-	-	-
CO4	3	3	-	-	-	-	-	-	-	-	-	-	-	-	-
CO5	3	3	-	-	-	-	-	-	-	-	-	-	-	-	-
AVG	3	3	-	-	-	-	-	-	-	-	-	-	-	-	-

1 - low, 2 - medium, 3 - high, ‘-' - no correlation

MA 242303 Transforms & Random Processes
UNIT – I Laplace Transform

Definition

If f(t) is a function of t defined for all t≥0, then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as the Laplace transform of f(t), provided the integral exists and it is denoted by L[f(t)] (or) F(s). $(i.e.)\quad L[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$

Sufficient Condition for the existence of Laplace Transform

If the function f(t) defined for t≥0 is

i) piecewise continuous in every finite interval in the range t≥0

and ii) of the exponential order, then L[f(t)] exists.

Note:

1) Laplace Transforms of all functions do not exist. For example, $L[\tan t]$ and $L[e^{t^2}]$ do not exist.
2) A function f(t) is said to be piecewise continuous in the finite interval a ≤ t ≤ b, if the interval can be divided into a finite number of sub-intervals such that
   - i) f(t) is continuous at every point inside each of the sub-intervals and
   - ii) f(t) has finite limits as t approaches the end points of each sub-interval from the interior of the sub-interval.
3) A function f(t) is said to be of the exponential order if $\lim_{t\to\infty} \{e^{-st} f(t)\} = finite$

Example: The function t^2 is of the exponential order.

For, $\lim_{t\to\infty} \{e^{-st} t^2\} = \lim_{t\to\infty} \frac{t^2}{e^{st}}$

$= \lim_{t\to\infty} \frac{2t}{s e^{st}}$ (Using L-Hospitals' Rule)

$= \lim_{t\to\infty} \frac{2}{s^2 e^{st}}$ (Again, using L-Hospitals' Rule)

$= \frac{2}{\infty} = 0$

= a finite number.

Linearity Property

$L[k_1 f(t) \pm k_2 g(t)] = k_1 L[f(t)] \pm k_2\, L[g(t)]$

$\Pr oof.\quad We\ \ have\ \ L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[k_1 f(t)\,\pm\,k_2 g(t)] = \int_0^\infty e^{-st}[k_1 f(t)\,\pm\,k_2 g(t)]\,dt$$

$$= k_1 \int_0^\infty e^{-st} f(t)\,dt \,\pm\, k_2 \int_0^\infty e^{-st} g(t)\ dt$$

$$= k_1 L[f(t)] \pm k_2\, L[g(t)]$$

Laplace Transform Formulas

1. $L(k) = \dfrac{k}{s},\ s > 0$

$$For,\ L[k] = \int_0^\infty e^{-st} k\,dt = k\int_0^\infty e^{-st}\,dt \ = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{-s}\left[e^{-\infty} - e^0\right]$$

$$= \frac{k}{-s}[0-1] = \frac{k}{s}$$

$e^{-\infty} = \dfrac{1}{e^{\infty}} = \dfrac{1}{\infty} = 0$

$e^0 = 1$

2. $L(e^{at}) = \dfrac{1}{s-a},\ s > a$

$$For,\ L[e^{at}] = \int_0^\infty e^{-st}\, e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt \ = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{1}{-(s-a)}\left[e^{-\infty} - e^0\right]$$

$$= \frac{1}{-(s-a)}[0-1] = \frac{1}{s-a}$$

3. $L(e^{-at}) = \dfrac{1}{s+a},\ s > -a$

$$For,\ L[e^{-at}] = \int_0^\infty e^{-st}\, e^{-at}\,dt = \int_0^\infty e^{-(s+a)t}\,dt \ = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{-(s+a)}\left[e^{-\infty} - e^0\right]$$

$$= \frac{1}{-(s+a)}[0-1] = \frac{1}{s+a}$$

4. $L(\sinh at) = \dfrac{a}{s^2 - a^2}$

$$For,\ \ L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right] = \frac{a}{s^2 - a^2}$$

5. $L(\cosh at) = \dfrac{s}{s^2 - a^2}$

$$For,\ \ L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right] = \frac{1}{2}\left[L(e^{at}) + L(e^{-at})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right] = \frac{s}{s^2 - a^2}$$

6. $L(\sin at) = \dfrac{a}{s^2 + a^2}$

$$For,\ \ L[\sin at] = L[I.P.\ of\ \ e^{iat}] = I.P.\,L[e^{iat}] = I.P.\frac{1}{s - ia}$$

$e^{i\theta} = \cos\theta + i\sin\theta$
$\cos\theta = R.P.\ of\ \ e^{i\theta}$
$\sin\theta = I.P.\ of\ \ e^{i\theta}$

$$= I.P.\left[\frac{s+ia}{(s-ia)(s+ia)}\right]$$

$$= I.P.\left[\frac{s+ia}{s^2 + a^2}\right] = \frac{a}{s^2 + a^2}$$

7. $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$For,\ \ L[\cos at] = L[R.P.\ of\ \ e^{iat}] = R.P.\,L[e^{iat}] = R.P.\frac{1}{s - ia}$$

$$= R.P.\left[\frac{s+ia}{(s-ia)(s+ia)}\right]$$

$$= R.P.\left[\frac{s+ia}{s^2 + a^2}\right] = \frac{s}{s^2 + a^2}$$

8. $L(t^n) = \dfrac{n!}{s^{n+1}}, \ s > 0 \ \& \ n > -1$

$$For, \ L[t^n] = \int_0^\infty e^{-st}\, t^n \, dt = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x}\, x^n \, dx$$

Put st = x
sdt = dx
dt = dx/s

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}$$

Gamma function

$$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx, \quad \Gamma(n+1) = \int_0^\infty e^{-x}\, x^n \, dx$$

$$\Gamma(n+1) = \begin{cases} n! & when \ n \ is \ \text{int}eger \\ n\Gamma(n) & when \ n \ is \ non-\text{int}eger \end{cases}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \begin{cases} \dfrac{n!}{s^{n+1}} & if \ n \ is \ \text{int}eger \\ \dfrac{n\Gamma(n)}{s^{n+1}} & if \ n \ is \ non-\text{int}eger \end{cases}$$

$$\Rightarrow L(t) = \frac{1}{s^2}, \ L(t^2) = \frac{2}{s^3}, \ L(t^3) = \frac{6}{s^4}, \ L(t^4) = \frac{24}{s^5} \ and \ so \ on.$$

Problems

1. *Find* $L(\sqrt{t}\,)$

$$Sol. \ L(\sqrt{t}\,) = L\left(t^{1/2}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2\, s^{3/2}}$$

2. *Find* $L\left(\dfrac{1}{\sqrt{t}}\right)$

$$Sol. \ L\left(\frac{1}{\sqrt{t}}\right) = L\left(t^{-1/2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

3. *Find* $L(t^{3/2})$

$$Sol. \ L(t^{3/2}) = \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} = \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{\frac{5}{2}}} = \frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{5}{2}}} = \frac{\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}} = \frac{3\sqrt{\pi}}{4s^{5/2}}$$

4. Find L(t^2 + 2t + 3)

Sol. $L(t^2+2t+3)=L(t^2)+2L(t)+L(3)$

$$=\frac{2}{s^3}+\frac{2}{s^2}+\frac{3}{s}=\frac{2+2s+3s^2}{s^3}$$

5. *Find* $L(\sin^2 2t)$

Sol. $L(\sin^2 2t)=L\left(\frac{1-\cos 4t}{2}\right)=\frac{1}{2}\left[L(1)-L(\cos 4t)\right]$

$$=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+16}\right]=\frac{1}{2}\left[\frac{s^2+16-s^2}{s\,(s^2+16)}\right]=\frac{8}{s\,(s^2+16)}$$

6. *Find* $L(\sin^3 2t)$

Sol. $L(\sin^3 2t)=L\left(\frac{3}{4}\sin 2t-\frac{1}{4}\sin 6t\right)$

$$=\frac{3}{4}L(\sin 2t)-\frac{1}{4}L(\sin 6t)$$

$$=\frac{3}{4}\left[\frac{2}{s^2+4}\right]-\frac{1}{4}\left[\frac{6}{s^2+36}\right]=\frac{3}{2(s^2+4)}-\frac{3}{2(s^2+36)}$$

$$=\frac{3(s^2+36)-3(s^2+4)}{2(s^2+4)\,(s^2+36)}$$

$$=\frac{108-12}{2(s^2+4)\,(s^2+36)}$$

$$=\frac{48}{(s^2+4)\,(s^2+36)}$$

7. *Find* $L(\sin 3t\cos t)$

Sol. $L(\sin 3t\cos t)=L\left[\frac{1}{2}(\sin 4t+\sin 2t)\right]=\frac{1}{2}[L(\sin 4t)+L(\sin 2t)]$

$$=\frac{1}{2}\left[\frac{4}{s^2+16}+\frac{2}{s^2+4}\right]$$

$$=\frac{1}{2}\left[\frac{4(s^2+4)+2(s^2+16)}{(s^2+16)(s^2+4)}\right]$$

$$=\frac{1}{2}\left[\frac{6s^2+48}{(s^2+16)(s^2+4)}\right]$$

$$=\frac{3s^2+24}{(s^2+16)(s^2+4)}.$$

8. *Find* $L(\cos 2t \cos t)$

Sol. $L(\cos 2t \cos t) = L\left[\frac{1}{2}(\cos 3t + \cos t)\right] = \frac{1}{2}[L(\cos 3t) + L(\cos t)]$

$$= \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{s(s^2+1) + s(s^2+9)}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{1}{2}\left[\frac{2s^3+10s}{(s^2+9)(s^2+1)}\right]$$

$$= \frac{s^3+5s}{(s^2+9)(s^2+1)}.$$

9. *Find* $L[f(t)]$ *where* $f(t) = \begin{cases} 0 & when \quad 0<t<2 \\ 3 & when \quad t>2 \end{cases}$

Sol. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^2 e^{-st}.0\, dt + \int_2^\infty e^{-st}.3\, dt$$

$$= 0 + 3\left[\frac{e^{-st}}{-s}\right]_2^\infty = \frac{3}{-s}[e^{-\infty} - e^{-2s}] = \frac{3}{-s}[0 - e^{-2s}] = \frac{3e^{-2s}}{s}$$

10. *Find* $L[f(t)]$ *where* $f(t) = \begin{cases} (t-1)^2 & when \quad t>1 \\ 0 & when \quad t<1 \end{cases}$

Sol. $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^1 e^{-st}.0\, dt + \int_1^\infty e^{-st}(t-1)^2\, dt$$

$$= 0 + \int_1^\infty (t-1)^2 e^{-st} dt$$

$$= \left[(t-1)^2\left(\frac{e^{-st}}{-s}\right) - 2(t-1)\left(\frac{e^{-st}}{s^2}\right) + 2\left(\frac{e^{-st}}{-s^3}\right)\right]_1^\infty$$

$$= \left[\{0\} - \left\{0 - 0 + \frac{2e^{-s}}{-s^3}\right\}\right] = \frac{2e^{-s}}{s^3}$$

Home Work

$1.\ Find\ \ L(\cos^2 3t)$ $\quad 2.\ Find\ \ L(\cos^3 4t)$ $\quad 3.\ Find\ \ L(\cos 5t \cos 2t)$

$4.\ Find\ \ L(\sin 2t \sin t)$ $\quad 5.\ Find\ \ L(\cos 3t \sin t)$

$$6.\ Find\ \ L[f(t)]\ \ where\ \ f(t) = \begin{cases} e^{-t} & when\ \ 0 < t < 4 \\ 0 & when\ \ t > 4 \end{cases}$$

$$7.\ Find\ \ L[f(t)]\ \ where\ \ f(t) = \begin{cases} \sin t & when\ \ 0 < t < \pi \\ 0 & when\ \ t > \pi \end{cases}$$

Laplace Transform of Derivatives

1. Prove that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$\Pr oof.\ \ We\ \ have\ \ L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\ e^{-st}(-s)\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L[f(t)]$$

$$= s\,L[f(t)] - f(0)$$

2. Prove that $L[f''(t)] = s^2\,L[f(t)] - sf(0) - f'(0)$

$$\Pr oof.\ \ We\ \ have\ \ L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore L[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty f'(t)\ e^{-st}(-s)\,dt$$

$$= [0 - f'(0)] + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\,L[f'(t)]$$

$$= s\,L[f'(t)] - f'(0)$$

$$= s\{sL[f(t) - f(0)\} - f'(0)$$

$$= s^2 L[f(t) - s\,f(0) - f'(0)$$

$$Similarly\ ,\quad L[f'''(t)] = s^3\, L[f(t)] - s^2 f(0) - s\, f'(0) - f''(0)$$

$$In\ \ general\,,$$

$$L[f^{(n)}(t)] = s^n\, L[f(t)] - s^{n-1} f(0) - s^{n-2}\, f'(0) - s^{n-3} f''(0) - .... - f^{(n-1)}(0)$$

Laplace Transform of some special functions

Unit step function

$$The\quad function\quad f(t) = \begin{cases} 0 & when\ \ t < a \\ 1 & when\ \ t > a \end{cases}$$

where a ≥ 0 is called Heavyside's Unit step function and is denoted by $u_a(t)$ or u(t – a).

$$In\ \ particular\,,\quad u_0(t) = \begin{cases} 0 & when\ \ t < 0 \\ 1 & when\ \ t > 0 \end{cases}$$

$$Now,\ \ L[u_a(t)] = \int_0^\infty e^{-st} u_a(t)\, dt$$

$$= \int_0^a e^{-st} .0\, dt + \int_a^\infty e^{-st} .1\, dt$$

$$= \ 0\ + \left[\frac{e^{-st}}{-s} \right]_a^\infty$$

$$= \frac{1}{-s} [0 - e^{-as}] = \frac{e^{-as}}{s}$$

$$In\ \ particular,\ \ L[u_0(t)] = \frac{e^0}{s} = \frac{1}{s}$$

Unit impulse function or Dirace Delta function

$$The\ \ function\ \ \delta_a(t)\ \ (or)\ \ \delta(t-a) = \lim_{h \to 0}\ f(t)$$

$$where\ \ f(t)\ \ is\ \ defined\ \ by\ \ f(t) = \begin{cases} \dfrac{1}{h} & when\ \ a \le t \le a+h \\ 0 & otherwise \end{cases}$$

is called Unit impulse function or Dirace Delta function.

$$Now,\ L[\delta_a(t)] = L\left[\lim_{h\to 0} f(t)\right] = \lim_{h\to 0} L[f(t)]$$

$$= \lim_{h\to 0} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{h\to 0} \int_a^{a+h} e^{-st}\,\frac{1}{h}\,dt$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\left(\frac{e^{-st}}{-s}\right)_a^{a+h}\right]$$

$$= \lim_{h\to 0}\left[\frac{e^{-s(a+h)} - e^{-sa}}{-sh}\right]$$

$$= e^{-as} \lim_{h\to 0}\left[\frac{1-e^{-sh}}{sh}\right]$$

$$= e^{-as} \lim_{h\to 0}\left[\frac{-e^{-sh}(-s)}{s}\right]$$

(by L-Hospitals' Rule)

$$= e^{-as} \lim_{h\to 0} e^{-sh}$$

$$= e^{-as}\,(1)\ = e^{-as}$$

$$In\ \ particular,\ L[\delta_0(t)] = e^0 = 1$$

Properties of Laplace Transform

1. Change of Scale property

If L[f(t)] = F(s), Prove that $(i)\ L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$ $\quad (ii)\ L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

Proof. We have $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$(i)\quad L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$$

put at = u $\Rightarrow$ adt = du

$$= \int_0^\infty e^{-s\frac{u}{a}} f(u)\,\frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u} f(u)\,du = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t} f(t)\,dt = \frac{1}{a}F\left(\frac{s}{a}\right)$$

$$(ii)\quad L\left[f\left(\frac{t}{a}\right)\right]=\int_0^\infty e^{-st} f\left(\frac{t}{a}\right) dt$$

put t/a = u
⇒ t = au
⇒dt = a du

$$=\int_0^\infty e^{-s\,au} f(u)\; a\,du$$

$$=a\int_0^\infty e^{-s\,a\,u} f(u)\; du = a\int_0^\infty e^{-(a\,s)\,t} f(t)\; dt = aF(a\,s)$$

2. First Shifting property

If L[f(t)] = F(s), Prove that $(i)\ L\,[e^{-at} f(t)] = F(s+a)\quad (ii)\ L\,[e^{at} f(t)] = F(s-a)$

Proof. We have $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$(i)\quad L\,[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)\; dt\; = \int_0^\infty e^{-(s+a)\,t} f(t)\; dt\; =\; F(s+a)$$

$$(ii)\quad L\,[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)\; dt\; = \int_0^\infty e^{-(s-a)\,t} f(t)\; dt\; =\; F(s-a)$$

3. Second Shifting property

If L[f(t)] = F(s), Prove that $L\,[f(t-a)u_a(t)] = e^{-as} F(s)$ **where 'a' is a positive constant and u$_a$(t) is the unit step function.**

Proof. We have $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$L[f(t-a)\,u_a(t)] = \int_0^\infty e^{-st} f(t-a)\,u_a(t)\,dt$$

$$=\int_0^a e^{-st} f(t-a).0\,dt + \int_a^\infty e^{-st} f(t-a).1\,dt$$

$$=\;0\;+\int_a^\infty e^{-st} f(t-a)\; dt$$

Put t – a = x
dt = dx
When t = a, $x = 0$
t = ∞, $x = \infty$

$$=\int_0^\infty e^{-s(x+a)} f(x)\; dx$$

$$=e^{-as}\int_0^\infty e^{-sx} f(x)\; dx = e^{-as}\int_0^\infty e^{-st} f(t)\; dt = e^{-as}F(s)$$

Problems

1. $Find\quad L(e^{-t}\cos 2t)$

$Sol.\quad L(\cos 2t)=\dfrac{s}{s^2+4}$

$$\therefore\ L(e^{-t}\cos 2t)=\frac{s+1}{(s+1)^2+4}=\frac{s+1}{s^2+2s+5}$$ (Using First shifting property)

2. $Find\quad L(e^{2t}\sinh 3t)$

$Sol.\quad L(\sinh 3t)=\dfrac{3}{s^2-9}$

$$\therefore\ L(e^{2t}\sinh 3t)=\frac{s-2}{(s-2)^2-9}=\frac{s-2}{s^2-4s-5}$$ (Using First shifting property)

3. *Find the Laplace transform of* $\sin t.u_{\pi}(t)$, *where* $u_{\pi}(t)$ *is the unit step function.*

$$Sol.\ L[\sin t.u_{\pi}(t)]=L[\sin(t-\pi+\pi).u_{\pi}(t)]$$
$$=-L[\sin(t-\pi).u_{\pi}(t)]$$
$$=-e^{-\pi s}L(\sin t)$$ (Using second shifting property)
$$=-\frac{e^{-\pi s}}{s^2+1}$$

Home Work

1. $Find\ L(e^{3t}\sin t)$
2. $Find\ L(e^{-2t}\cosh 3t)$
3. $Find\ L(\sinh 3t\cos^2 t)$
4. $Find\ L(e^{-4t}\sin 3t\cos 2t)$

Theorem: Prove that $L\left[\int_0^t f(t)\,dt\right]=\dfrac{1}{s}L[f(t)]$

$\Pr oof.\quad Let \int_0^t f(t)\,dt=F(t).\ Then\ F'(t)=f(t)\ and\ f(0)=0$

$We\ have\ L[F'(t)]=s\,L[F(t)]-F(0)$

$$L[f(t)]=s\,L\left[\int_0^t f(t)\,dt\right]-0$$

$$(i.e.)\ L\left[\int_0^t f(t)\,dt\right]=\frac{1}{s}L[f(t)]$$

Theorem:

If $L[f(t)] = F(s)$ *then* $L\left[t\, f(t)\right] = -\dfrac{d}{ds}F(s)$ (or) $-\dfrac{d}{ds}L[f(t)]$. *Deduce* $L[t^n f(t)]$

Pr*oof*. *By definition*, $F(s) = \int\limits_0^\infty e^{-st} f(t)\, dt = L[f(t)]$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int\limits_0^\infty e^{-st} f(t)\, dt$$

$$\frac{d}{ds}F(s) = \int\limits_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]\, dt$$

$$= \int\limits_0^\infty e^{-st}(-t) f(t)\, dt$$

$$= -\int\limits_0^\infty e^{-st}\, t\, f(t)\, dt = -L[t\, f(t)]$$

$$(i.e.)\ L[t\, f(t)] = -\frac{d}{ds}F(s)$$

To find $L[t^n f(t)]$

$$L[t^2 f(t)] = L[t.tf(t)]$$

$$= -\frac{d}{ds}L[t\, f(t)]$$

$$= -\frac{d}{ds}.-\frac{d}{ds}L[f(t)]$$

$$L[t^2 f(t)] = (-1)^2\frac{d^2}{ds^2}L[f(t)]$$

Similarly, $L[t^3 f(t)] = (-1)^3\dfrac{d^3}{ds^3}L[f(t)]$

Pr*oceeding like this*, *we get*

$$L[t^n f(t)] = (-1)^n\frac{d^n}{ds^n}L[f(t)]$$

Theorem:

If $L[f(t)] = F(s)$ *and if* $\dfrac{f(t)}{t}$ *has a* lim*it as* $t \to 0$, *then*

$$L\left[\frac{f(t)}{t}\right] = \int\limits_s^\infty F(s)\, ds\ (or)\ \int\limits_s^\infty L[f(t)]\, ds$$

$$\Pr oof.\ \ By\ \ definition,\ \ F(s)=\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt \ = \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$\int_s^\infty F(s)\,ds = \int_0^\infty \left[0-\frac{e^{-st}}{-t}\right] f(t)\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{f(t)}{t}\,dt = L\left[\frac{f(t)}{t}\right]$$

Note:

$$L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty\int_s^\infty F(s)\,ds\,ds\ \ (or)\ \ \int_s^\infty\int_s^\infty L[f(t)]\,ds\,ds$$

Problems

1. *Find* $L(t\,e^{-a\,t})$

$$Sol.\ \ L(t\,e^{-a\,t}) = -\frac{d}{ds}L(e^{-at}) = -\frac{d}{ds}\left(\frac{1}{s+a}\right) = -\left[\frac{-1}{(s+a)^2}\right] = \frac{1}{(s+a)^2}$$

2. *Find* $L(t^2\,e^{-3\,t})$

$$Sol.\ \ L(t^2\,e^{-3\,t}) = (-1)^2\,\frac{d^2}{ds^2}L(e^{-3t})$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) = \frac{d}{ds}\left[\frac{-1}{(s+3)^2}\right] = \frac{-(-2)}{(s+3)^3} = \frac{2}{(s+3)^3}$$

3. *Find* $L(t\,e^{-t}\sin t)$

$$Sol.\ \ L(t\,e^{-t}\sin t) = -\frac{d}{ds}L(e^{-t}\sin t)$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+2s+2}\right)$$

$$= -\left[\frac{-(2s+2)}{(s^2+2s+2)^2}\right]$$

$$= \frac{2(s+1)}{(s^2+2s+2)^2}$$

$$L(\sin t) = \frac{1}{s^2+1}$$

$$L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$$

$$= \frac{1}{s^2+2s+2}$$

4. *Find the Laplace transform of* $e^{-t}\int_0^t t\cos t\,dt$

Sol. $L\left[e^{-t}\int_0^t t\cos t\,dt\right] = ?$

$$L\left[\int_0^t t\cos t\,dt\right] = \frac{1}{s}L(t\cos t)$$

$$= \frac{1}{s}\left[-\frac{d}{ds}L(\cos t)\right]$$

$$= -\frac{1}{s}\left[\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{1}{s}\left[\frac{(s^2+1).1-s(2s)}{(s^2+1)^2}\right] = -\frac{1}{s}\left[\frac{-s^2+1}{(s^2+1)^2}\right]$$

$$(i.e.)\ L\left[\int_0^t t\cos t\,dt\right] = \frac{1}{s}\left[\frac{s^2-1}{(s^2+1)^2}\right]$$

$$\therefore L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-1}{\{(s+1)^2+1\}^2}\right]$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

5. Does Laplace transform of $\frac{\cos at}{t}$ exist ? Justify.

Sol. $\lim_{t\to 0}\frac{f(t)}{t} = \lim_{t\to 0}\frac{\cos at}{t} = \frac{\cos 0}{0} = \frac{1}{0} = \infty$

Hence $L\left[\frac{\cos at}{t}\right]$ does not exists.

6. *Find* $L\left(\frac{1-e^{-2t}}{t}\right)$

Sol. $L\left(\frac{1-e^{-2t}}{t}\right) = \int_s^\infty L(1-e^{-2t})ds = \int_s^\infty \left[L(1)-L(e^{-2t})\right]ds$

$$= \int_s^\infty \left[\frac{1}{s}-\frac{1}{s+2}\right]ds$$

$$= \left[\log s - \log(s+2)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s+2}\right)\right]_s^\infty$$

$$= -\left[\log\left(\frac{s+2}{s}\right)\right]_s^\infty = -\left[\log\left(1+\frac{2}{s}\right)\right]_s^\infty$$

$$= -\left[\log(1+0) - \log\left(1+\frac{2}{s}\right)\right]$$

$$= -\left[0 - \log\left(\frac{s+2}{s}\right)\right] \qquad (\log 1 = 0)$$

$$= \log\left(\frac{s+2}{s}\right)$$

7. *Find* $L\left[e^{2t}\int_0^t \frac{\sin 3t}{t}\,dt\right]$

Sol. $L\left[\int_0^t \frac{\sin 3t}{t}\,dt\right] = \frac{1}{s}L\left(\frac{\sin 3t}{t}\right) = \frac{1}{s}\int_s^\infty L(\sin 3t)\,ds$

$$= \frac{1}{s}\int_s^\infty \frac{3}{s^2+9}\,ds$$

$$= \frac{3}{s}\left[\frac{1}{3}\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty = \frac{1}{s}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{3}\right)\right]$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right)\right]$$

$$= \frac{1}{s}\cot^{-1}\left(\frac{s}{3}\right)$$

$$\therefore L\left[e^{2t}\int_0^t \frac{\sin 3t}{t}\,dt\right] = \frac{1}{s-2}\cot^{-1}\left(\frac{s-2}{3}\right)$$

8. *Find* $L\left[\frac{\sin^2 t}{t}\right]$

Sol. $L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L(\sin^2 t)\,ds = \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)ds$

$$= \frac{1}{2}\int_s^{\infty}[L(1) - L(\cos 2t)]\,ds$$

$$= \frac{1}{2}\int_s^{\infty}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^{\infty}$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^{\infty}$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

9. Find L[t^2 2^t].

Sol. L[t^2 2^t] = $(-1)^2 \dfrac{d^2}{ds^2} L[2^t]$ $\qquad (\because a^x = e^{x\log a})$

$$= \frac{d^2}{ds^2}L[e^{t\log 2}] = \frac{d^2}{ds^2}\left[\frac{1}{s-\log 2}\right]$$

$$= \frac{d}{ds}\left[\frac{-1}{(s-\log 2)^2}\right]$$

$$= \frac{2}{(s-\log 2)^3}.$$

10. Find $L\left(\dfrac{\sin at}{t}\right)$

Sol. $L\left(\dfrac{\sin at}{t}\right) = \displaystyle\int_s^{\infty} L(\sin at)ds = \int_s^{\infty}\frac{a}{s^2+a^2}ds = a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^{\infty}$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right).$$

11. *Find* $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Sol. $$L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^\infty L(e^{-at}-e^{-bt})ds=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$=\log\left(\frac{s+b}{s+a}\right).$$

12. *Find* $L\left[\frac{e^{at}-\cos bt}{t}\right]$

Sol. $$L\left[\frac{e^{at}-\cos bt}{t}\right]=\int_s^\infty L(e^{at}-\cos bt)\,ds=\int_s^\infty\left(\frac{1}{s-a}-\frac{s}{s^2+b^2}\right)ds$$

$$=\left[\log(s-a)-\frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$=\left[\frac{1}{2}\log(s-a)^2-\frac{1}{2}\log(s^2+b^2)\right]$$

$$=\frac{1}{2}\left[\log\left(\frac{(s-a)^2}{s^2+b^2}\right)\right]_s^\infty$$

$$=\frac{1}{2}\log\left(\frac{s^2+b^2}{(s-a)^2}\right)$$

13. *If* $L[\sin\sqrt{t}\,]=\frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}$ *Find* $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

Sol. $$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]=2L\left[\frac{d}{dt}\sin\sqrt{t}\right]$$

$$=2\left\{sL\left[\sin\sqrt{t}\right]-f(0)\right\}$$

$$=2\left[s.\frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}-0\right]=s.\frac{\sqrt{\pi}}{s\sqrt{s}}e^{-\frac{1}{4s}}=\sqrt{\frac{\pi}{s}}\;e^{-\frac{1}{4s}}$$

Home Work

1\. $Find\ L(t\sin at)$ 2. $Find\ L(t\cos 2t)$ 3. $Find\ L(t^2\sin 5t)$

4\. $Find\ L(t\cos^2 t)$ 5. $Find\ L(t\,e^{-t}\cos t)$ 6. $Find\ L(t\,e^{-2t}\sin 3t)$

7\. $Find\ L(t^2e^{-3t}\sin 2t)$ 8. $Find\ L(t^2e^{2t}\cos 2t)$ 9. $Find\ L(\sin 2t-2t\cos 2t)$

10\. $Find\ L(t\cosh t\sin 2t)$ 11. $Find\ L\left[\int_0^t t\,e^{-t}\sin t\,dt\right]$ 12. $Find\ L\left(\frac{1-\cos t}{t}\right)$

13\. $Find\ L\left(\frac{e^{-t}-e^{-2t}}{t}\right)$ 14. $Find\ L\left(\frac{\cos 2t-\cos 3t}{t}\right)$ 15. $Find\ L\left(\frac{e^{-3t}\sin 2t}{t}\right)$

16\. $Find\ L\left(\frac{1-\cos t}{t^2}\right)$ 17. $Find\ L\left[e^{-t}\int_0^t\frac{\sin t}{t}\,dt\right]$

18\. $Find\ L\left\{\left[t\int_0^t e^{-4t}\cos 3t\,dt\right]+\frac{\sin 5t}{t}\right\}$

Using Laplace Transform we can evaluate certain integrals
Problems

1. Using *Laplace transform evaluate* $\int_0^\infty e^{-2t}\sin 3tdt$

$Sol.\ \ By\ \ definition,\ \int_0^\infty e^{-s\,t}\sin 3tdt = L(\sin 3t)$

$$=\frac{3}{s^2+9}$$

$put\ \ s=2,\ we\ \ get$

$$\int_0^\infty e^{-2\,t}\sin 3tdt=\frac{3}{2^2+9}=\frac{3}{13}$$

2. Using *Laplace transform evaluate* $\int_0^\infty te^{-3t}\sin 2tdt$

Sol. By definition, $\int_0^\infty t\,e^{-s\,t}\sin 2t\,dt = L(t\sin 2t)$

$$=-\frac{d}{ds}L[\sin 2t]$$

$$= -\frac{d}{ds}.\left(\frac{2}{s^2+4}\right)$$

$$= -\left[\frac{-2}{(s^2+4)^2}.2s\right]$$

$$= \frac{4s}{(s^2+4)^2}.$$

put s = 3, we get

$$\int_0^\infty t\, e^{-3t} \sin 2t\, dt = \frac{4(3)}{(9+4)^2} = \frac{12}{169}.$$

3. Using Laplace transform evaluate $\int_0^\infty e^{-2t}\frac{\sin^2 t}{t}dt$

Sol. By definition, $\int_0^\infty e^{-st}\frac{\sin^2 t}{t}dt = L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty L(\sin^2 t)ds$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

put s = 2, we get

$$\int_0^\infty e^{-2t}\frac{\sin^2 t}{t}dt \quad = \frac{1}{4}\log\left(\frac{4+4}{4}\right) = \frac{1}{4}\log 2.$$

4. Using Laplace transform evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$

Sol. By definition, $\int_0^\infty e^{-st}\,\frac{e^{-t}-e^{-2t}}{t}dt = L\left(\frac{e^{-t}-e^{-2t}}{t}\right) = \int_s^\infty L(e^{-t}-e^{-2t})ds$

$$= \int_s^\infty \left(\frac{1}{s+1}-\frac{1}{s+2}\right)ds$$

$$= [\log(s+1)-\log(s+2)]_s^\infty$$

$$= \left[\log\left(\frac{s+1}{s+2}\right)\right]_s^\infty$$

$$= \log\left(\frac{s+2}{s+1}\right)$$

put s = 0, we get

$$\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt = \log\left(\frac{0+2}{0+1}\right) = \log 2$$

Home Work

1. *Evaluate* $\int_0^\infty t\,e^{-3t}\cos t\;dt$ 2. *Evaluate* $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt$

3. *Evaluate* $\int_0^\infty \frac{e^{-3t}-e^{-6t}}{t}dt$

Initial value theorem

If L[f(t)] = F(s) then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof. We have $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= s\,F(s) - f(0)$$

Taking limits on both sides as $s \to \infty$, we get

$$\lim_{s\to\infty}[sF(s)-f(0)] = \lim_{s\to\infty} L[f'(t)]$$

$$= \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt$$

$$= 0$$

$$\lim_{s\to\infty} sF(s) = f(0)$$

$$(i.e.)\ \lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$$

Final value theorem

If L[f(t)] = F(s) then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof. We have $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= s\,F(s) - f(0)$$

Taking limits on both sides as $s \to 0$, we get

$$\lim_{s\to 0}[sF(s) - f(0)] = \lim_{s\to 0} L[f'(t)]$$

$$= \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt$$

$$= \left[f(t)\right]_0^\infty$$

$$\Rightarrow \lim_{s\to 0} sF(s) - f(0) = f(\infty) - f(0)$$

$$(i.e.)\ \lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

Problems

1. Verify initial and final value theorem for f(t) = 1 + e^{-t} (sint + cost).

Sol. Given f(t) = 1 + e^{-t} (sint + cost)

F(s) = L[f(t)] = L[1 + e^{-t} (sint + cost)]

$$F(s) = L(1) + L(e^{-t}\sin t) + L(e^{-t}\cos t)$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$$

$$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{s^2+2s+2}$$

$$L(\sin t) = \frac{1}{s^2+1}$$

$$L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2}$$

$$L(\cos t) = \frac{s}{s^2+1}$$

$$L(e^{-t}\cos t) = \frac{s+1}{(s+1)^2+1} = \frac{s+1}{s^2+2s+2}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(1 + e^{-t}\sin t + e^{-t}\cos t)$$

$$= 1 + 0 + 1 = 2$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{s^2+2s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{1}{s+2+2/s} + \frac{1+1/s}{1+2/s+2/s^2}\right]$$

$$= 1 + 0 + 1 = 2$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Hence initial value theorem is verified.

$$Now, \quad \lim_{t\to\infty} f(t) = \lim_{t\to\infty} (1 + e^{-t}\sin t + e^{-t}\cos t)$$

$$= 1 + 0 + 0 = 1$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{s}{s^2+2s+2} + \frac{s(s+1)}{s^2+2s+2}\right]$$

$$= 1 + 0 + 0 = 1$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Hence final value theorem is verified.

2. Verify the initial value theorem for the function f(t) = 1 + e^{-2t}

Sol. Given f(t) = 1 + e^{-2t}

F(s) = L[f(t)] = L[1 + e^{-2t}]

$$F(s) = L(1) + L(e^{-2t})$$

$$F(s) = \frac{1}{s} + \frac{1}{s+2}$$

$$sF(s) = 1 + \frac{s}{s+2}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(1 + e^{-2t})$$

$$= 1 + 1 = 2$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s}{s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{1}{1+2/s}\right]$$

$$= 1 + 1 = 2$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Hence initial value theorem is verified.

Home Work

1. Verify initial and final value theorem for $f(t) = t^2 e^{-3t}$

Periodic function

A function f(t) is said to be a periodic function, if there exists a constant P(> 0) such that f(t + P) = f(t) for all values of t. The least value of P is called the period of the function.

Example: (i) f(t) = sint

= sin(2π + t) = sin(4π + t) = …….

The function has periods 2π, 4π, etc. However 2π is the least value and therefore 2π is the period of sint.

(ii) f(t) = cost

= cos(2π + t) = cos(4π + t) = …….

The function has periods 2π, 4π, etc. However 2π is the least value and therefore 2π is the period of cost.

(iii) f(t) = tant

= tan(π + t) = tan(2π + t) = …….

The function has periods π, 2π, etc. However π is the least value and therefore π is the period of tant.

(iv) f(t) = | sint |

= | sin(π + t) | = | sin(2π + t) | = …….

The function has periods π, 2π, etc. However π is the least value and therefore π is the period of | sint |.

Laplace transform of f(t) which is periodic with period 'p' is

$$L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt$$

Problems

1. Find the Laplace transform of the square wave given by

$$f(t) = \begin{cases} E, & 0 \le t \le \frac{T}{2} \\ -E, & \frac{T}{2} \le t \le T \end{cases} \quad and \quad f(t+T) = f(t)$$

Sol. The given function is periodic with period 'T'.

$$\therefore L[f(t)] = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-Ts}}\left[\int_0^{T/2} e^{-st}.Edt + \int_{T/2}^{T} e^{-st}(-E)dt\right]$$

$$=\frac{E}{1-e^{-Ts}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{T/2}-\left(\frac{e^{-st}}{-s}\right)_{T/2}^{T}\right]$$

$$=\frac{E}{-s(1-e^{-Ts})}\left[(e^{-sT/2}-1)-(e^{-sT}-e^{-sT/2})\right]$$

$$=\frac{E(2e^{-sT/2}-1-e^{-sT})}{-s(1-e^{-Ts})}$$

$$=\frac{E(1-2e^{-sT/2}+e^{-sT})}{s(1-e^{-Ts})}$$

$$=\frac{E(1-e^{-sT/2})^2}{s(1-e^{-sT/2})(1+e^{-sT/2})}$$

$$=\frac{E(1-e^{-sT/2})}{s(1+e^{-sT/2})} \qquad \because \tanh\left(\frac{x}{2}\right)=\frac{e^x-1}{e^x+1}$$

$$=\frac{E\left(1-\frac{1}{e^{sT/2}}\right)}{s\left(1+\frac{1}{e^{sT/2}}\right)}=\frac{E(e^{sT/2}-1)}{s(e^{sT/2}+1)}=\frac{E}{s}\tanh\left(\frac{sT}{4}\right)$$

2. Find the Laplace transform of the 'meoander function' defined as f(t) = 1 when $0 < t < \frac{a}{2}$ and f(t) = –1 when $\frac{a}{2} < t < a$ and f(t) is periodic with period 'a' so that f(t + a) = f(t) for t > 0.

Sol. The given function is periodic with period 'a'.

$$\therefore L[f(t)]=\frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}.1dt+\int_{a/2}^{a} e^{-st}(-1)dt\right]$$

$$=\frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2}-\left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$=\frac{1}{-s(1-e^{-as})}\left[(e^{-sa/2}-1)-(e^{-sa}-e^{-sa/2})\right]$$

$$=\frac{2e^{-sa/2}-1-e^{-sa}}{-s(1-e^{-as})}$$

$$=\frac{1-2e^{-sa/2}+e^{-sa}}{s(1-e^{-as})}$$

$$=\frac{(1-e^{-sa/2})^2}{s(1-e^{-as/2})(1+e^{-sa/2})}$$

$$\because \tanh\left(\frac{x}{2}\right)=\frac{e^x-1}{e^x+1}$$

$$=\frac{(1-e^{-sa/2})}{s(1+e^{-sa/2})}=\frac{(e^{sa/2}-1)}{s(e^{sa/2}+1)}$$

$$=\frac{1}{s}\tanh\left(\frac{sa}{4}\right).$$

3. Prove that the Laplace transform of the rectangular wave of period 2π defined by

$$f(t)=\begin{cases} t, & 0\le t\le \pi \\ 2\pi-t, & \pi\le t\le 2\pi \end{cases} \quad is \quad \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right).$$

Sol. The given function is periodic with period '2π'.

$$\therefore L[f(t)]=\frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} e^{-st}.t\,dt+\int_{\pi}^{2\pi} e^{-st}(2\pi-t)dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\left[t.\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^{\pi}+\left[(2\pi-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\left\{-\frac{\pi}{s}e^{-s\pi}-\frac{e^{-s\pi}}{s^2}\right\}-\left\{0-\frac{1}{s^2}\right\}+\left\{0+\frac{e^{-s2\pi}}{s^2}\right\}-\left\{\frac{\pi e^{-s\pi}}{-s}+\frac{e^{-s\pi}}{s^2}\right\}\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\frac{1}{s^2}-\frac{2e^{-\pi s}}{s^2}+\frac{e^{-2\pi s}}{s^2}\right]$$

$$=\frac{1}{1-e^{-2\pi s}}.\frac{(1-e^{-\pi s})^2}{s^2}$$

$$=\frac{(1-e^{-\pi s})^2}{s^2(1-e^{-\pi s})(1+e^{-\pi s})}$$

$$=\frac{(1-e^{-\pi s})}{s^2(1+e^{-\pi s})}=\frac{(e^{\pi s}-1)}{s^2(e^{\pi s}+1)}=\frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right).$$

4. Find the Laplace transform of | sint |.

Sol. π is the period of | sint |. The given function is a periodic function with period ‘π’.

$$\therefore L[f(t)] = \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st}\,|\sin t|\,dt$$

$$= \frac{1}{1-e^{-\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt$$

$$= \frac{1}{1-e^{-\pi s}}\left[\frac{e^{-st}}{s^2+1}(-s.\sin t - 1.\cos t)\right]_0^{\pi}$$

$$= \frac{1}{1-e^{-\pi s}}\left[\left\{\frac{e^{-s\pi}}{s^2+1}(0+1)\right\} - \left\{\frac{1}{s^2+1}(0-1)\right\}\right]$$

$$= \frac{1}{1-e^{-\pi s}}\,\frac{e^{-s\pi}+1}{s^2+1}$$

$$= \frac{1}{(s^2+1)}\,\frac{e^{\pi s}+1}{e^{\pi s}-1}$$

$$= \frac{1}{(s^2+1)}\coth\left(\frac{\pi s}{2}\right)$$

5. Find the Laplace transform of f(t) if f(t) = e^t, 0 < t < 2π.

Sol. The given function is a periodic function with period ‘2π’.

$$\therefore L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}.e^{t}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-(s-1)t}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^{2\pi}$$

$$= \frac{-1}{(1-e^{-2\pi s})(s-1)}\left[e^{-(s-1)2\pi} - 1\right]$$

$$= \frac{1-e^{-2\pi(s-1)}}{(s-1)(1-e^{-2\pi s})}$$

6. Find the Laplace transform of the function given by

$$f(t)=\begin{cases}\dfrac{4Et}{T}-E, & 0\le t\le \dfrac{T}{2}\\ 3E-\dfrac{4Et}{T}, & \dfrac{T}{2}\le t\le T\end{cases} \quad and\ f(t+T)=f(t).$$

Sol. The given function is periodic with period 'T '.

$$\therefore L[f(t)]=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-Ts}}\left[\int_0^{T/2} e^{-st}\left(\frac{4Et}{T}-E\right)dt+\int_{T/2}^{T} e^{-st}\left(3E-\frac{4Et}{T}\right)dt\right]$$

$$=\frac{1}{1-e^{-Ts}}\left\{\left[\left(\frac{4Et}{T}-E\right)\left(\frac{e^{-st}}{-s}\right)-\left(\frac{4E}{T}\right)\left(\frac{e^{-st}}{s^2}\right)\right]_0^{T/2}\right.$$

$$\left.+\left[\left(3E-\frac{4Et}{T}\right)\left(\frac{e^{-st}}{-s}\right)-\left(\frac{-4E}{T}\right)\left(\frac{e^{-st}}{s^2}\right)\right]_{T/2}^{T}\right\}$$

$$=\frac{1}{1-e^{-Ts}}\left\{\left[\left(\frac{-E}{s}e^{-sT/2}-\frac{4E}{Ts^2}e^{-sT/2}\right)-\left(\frac{E}{s}-\frac{4E}{Ts^2}\right)\right]\right.$$

$$\left.+\left[\left(\frac{E}{s}e^{-sT}+\frac{4E}{Ts^2}e^{-sT}\right)-\left(\frac{-E}{s}e^{-sT/2}+\frac{4E}{Ts^2}e^{-sT/2}\right)\right]\right\}$$

$$=\frac{1}{1-e^{-Ts}}\left[\frac{4E}{Ts^2}-\frac{E}{s}-\frac{8E}{Ts^2}e^{-sT/2}+\frac{E}{s}e^{-sT}+\frac{4E}{Ts^2}e^{-sT}\right]$$

$$=\frac{1}{1-e^{-Ts}}\left[\frac{4E}{Ts^2}\left(1-2e^{-sT/2}+e^{-sT}\right)-\frac{E}{s}\left(1-e^{-sT}\right)\right]$$

$$=\frac{1}{1-e^{-Ts}}\left[\frac{4E}{Ts^2}\left(1-e^{-sT/2}\right)^2-\frac{E}{s}\left(1-e^{-sT}\right)\right]$$

$$=\frac{4E\left(1-e^{-sT/2}\right)^2}{Ts^2(1-e^{-Ts})}-\frac{E}{s}$$

$$=\frac{4E\left(1-e^{-sT/2}\right)^2}{Ts^2(1-e^{-sT/2})(1+e^{-sT/2})}-\frac{E}{s}$$

$$=\frac{4E\left(1-e^{-sT/2}\right)}{Ts^2(1+e^{-sT/2})}-\frac{E}{s}=\frac{4E\left(e^{sT/2}-1\right)}{Ts^2(e^{sT/2}+1)}-\frac{E}{s}$$

$$=\frac{4E}{Ts^2}\tanh\left(\frac{sT}{4}\right)-\frac{E}{s}.$$

7. Find the Laplace transform of the function f(t) with period $\frac{2\pi}{w}$ for

$$f(t) = \begin{cases} \sin wt, & 0 < t < \pi / w \\ 0, & \pi / w < t < 2\pi / w \end{cases}$$

Sol. The given function is periodic with period $\frac{2\pi}{w}$.

$$\therefore L[f(t)] = \frac{1}{1 - e^{-2\pi s / w}} \int_0^{2\pi / w} e^{-st} f(t) dt$$

$$= \frac{1}{1 - e^{-2\pi s / w}} \left[\int_0^{\pi / w} e^{-st}.\sin wt\, dt + \int_{\pi / w}^{2\pi / w} e^{-st} (0) dt \right]$$

$$= \frac{1}{1 - e^{-2\pi s / w}} \int_0^{\pi / w} e^{-st}.\sin wt\, dt$$

$$= \frac{1}{1 - e^{-2\pi s / w}} \left[\frac{e^{-st}}{s^2 + w^2} (-s.\sin wt - w.\cos wt) \right]_0^{\pi / w}$$

$$= \frac{1}{1 - e^{-2\pi s / w}} \left[\left\{ \frac{e^{-s\pi / w}}{s^2 + w^2} (0 + w) \right\} - \left\{ \frac{1}{s^2 + w^2} (0 - w) \right\} \right]$$

$$= \frac{1}{1 - e^{-2\pi s / w}} \frac{w + we^{-s\pi / w}}{s^2 + w^2}$$

$$= \frac{w(1 + e^{-s\pi / w})}{(s^2 + w^2)(1 + e^{-s\pi / w})(1 - e^{-s\pi / w})}$$

$$= \frac{w}{(s^2 + w^2)(1 - e^{-\pi s / w})}.$$

Home Work

1. Find the Laplace transform of the rectangular wave given by

$$f(t) = \begin{cases} 1, & 0 \le t \le b \\ -1, & b \le t \le 2b \end{cases}$$

Inverse Laplace Transform

1. $L(k)=\frac{k}{s} \Rightarrow L^{-1}\left(\frac{k}{s}\right)=k$

2. $L(e^{at})=\frac{1}{s-a} \Rightarrow L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$

3. $L(e^{-at})=\frac{1}{s+a} \Rightarrow L^{-1}\left[\frac{1}{s+a}\right]=e^{-at}$

4. $L(\sinh at)=\frac{a}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{a}{s^2-a^2}\right]=\sinh at$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2-a^2}\right]=\frac{\sinh at}{a}$$

5. $L(\cosh at)=\frac{s}{s^2-a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$

6. $L(\sin at)=\frac{a}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sin at$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{\sin at}{a}$$

7. $L(\cos at)=\frac{s}{s^2+a^2} \Rightarrow L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at$

8. $L(t^n)=\frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left[\frac{n!}{s^{n+1}}\right]=t^n$

$$\Rightarrow L^{-1}\left[\frac{1}{s^{n+1}}\right]=\frac{t^n}{n!}$$

$n=1 \quad \Rightarrow L^{-1}\left[\frac{1}{s^2}\right]=t$

$n=2 \quad \Rightarrow L^{-1}\left[\frac{1}{s^3}\right]=\frac{t^2}{2}$

$n=3 \quad \Rightarrow L^{-1}\left[\frac{1}{s^4}\right]=\frac{t^3}{6}$

$n=4 \quad \Rightarrow L^{-1}\left[\frac{1}{s^5}\right]=\frac{t^4}{24}$ and so on.

Type – 1

$$If\ L[f(t)] = F(s),\ then\ L\,[e^{-at} f(t)] = F(s+a)$$ (by First shifting property)

$$\Rightarrow L^{-1}\left[F(s+a)\right] = e^{-at} f(t)$$

$$\Rightarrow L^{-1}\left[F(s+a)\right] = e^{-at}\, L^{-1}\left[F(s)\right]$$

$$Similarly,\ L^{-1}\left[F(s-a)\right] = e^{at}\, L^{-1}\left[F(s)\right]$$

Problems

1. $Find\ L^{-1}\left[\dfrac{1}{(s+a)^2}\right]$

$$Sol.\ L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at} L^{-1}\left[\frac{1}{s^2}\right] = e^{-at}.t = t\,e^{-at}$$

2. $Find\ L^{-1}\left[\dfrac{1}{(s-2)^2+16}\right]$

$$Sol.\ L^{-1}\left[\frac{1}{(s-2)^2+16}\right] = e^{2t} L^{-1}\left[\frac{1}{s^2+16}\right] = \frac{e^{2t}\sin 4t}{4}$$

3. $Find\ L^{-1}\left[\dfrac{s+3}{(s+3)^2+9}\right]$

$$Sol.\ L^{-1}\left[\frac{s+3}{(s+3)^2+9}\right] = e^{-3t} L^{-1}\left[\frac{s}{s^2+9}\right] = e^{-3t}\cos 3t$$

4. $Find\ L^{-1}\left[\dfrac{s}{s^2+2s+5}\right]$

$$Sol.\ L^{-1}\left[\frac{s}{s^2+2s+5}\right] = L^{-1}\left[\frac{s}{(s+1)^2+4}\right]$$

$$= L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+4}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+4}\right] - L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= e^{-t}\, L^{-1}\left[\frac{s}{s^2+4}\right] - e^{-t} L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= e^{-t}\cos 2t - e^{-t}\,\frac{\sin 2t}{2}$$

$$= \frac{e^{-t}}{2}(2\cos 2t - \sin 2t)$$

5. *Find* $L^{-1}\left[\dfrac{s}{s^2-4s+20}\right]$

Sol. $L^{-1}\left[\dfrac{s}{s^2-4s+20}\right]=L^{-1}\left[\dfrac{s}{(s-2)^2+16}\right]$

$$=L^{-1}\left[\frac{(s-2)+2}{(s-2)^2+16}\right]$$

$$=L^{-1}\left[\frac{s-2}{(s-2)^2+16}\right]+2\,L^{-1}\left[\frac{1}{(s-2)^2+16}\right]$$

$$=e^{2t}\,L^{-1}\left[\frac{s}{s^2+16}\right]+2\,e^{2t}L^{-1}\left[\frac{1}{s^2+16}\right]$$

$$=e^{2t}\cos 4t+2\,e^{2t}\,\frac{\sin 4t}{4}$$

$$=\frac{e^{2t}}{2}(2\cos 4t+\sin 4t)$$

6. *Find* $L^{-1}\left[\dfrac{s-1}{s^2+4s-12}\right]$

Sol. $L^{-1}\left[\dfrac{s-1}{s^2+4s-12}\right]=L^{-1}\left[\dfrac{s-1}{(s+2)^2-16}\right]$

$$=L^{-1}\left[\frac{(s+2)-3}{(s+2)^2-16}\right]$$

$$=L^{-1}\left[\frac{s+2}{(s+2)^2-16}\right]-3\,L^{-1}\left[\frac{1}{(s+2)^2-16}\right]$$

$$=e^{-2t}\,L^{-1}\left[\frac{s}{s^2-16}\right]-3\,e^{-2t}L^{-1}\left[\frac{1}{s^2-16}\right]$$

$$=e^{-2t}\cosh 4t-3e^{-2t}\,\frac{\sinh 4t}{4}$$

$$=\frac{e^{-2t}}{4}(4\cosh 4t-3\sinh 4t)$$

7. *Find* $L^{-1}\left[\dfrac{s-3}{s^2+4s+13}\right]$

Sol. $L^{-1}\left[\dfrac{s-3}{s^2+4s+13}\right]=L^{-1}\left[\dfrac{s-3}{(s+2)^2+9}\right]$

$$=L^{-1}\left[\frac{(s+2)-5}{(s+2)^2+9}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] - 5\,L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}\,L^{-1}\left[\frac{s}{s^2+9}\right] - 5\,e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-2t}\cos 3t - 5\,e^{-2t}\,\frac{\sin 3t}{3}$$

$$= \frac{e^{-2t}}{3}(3\cos 3t - 5\sin 3t)$$

Home Work

Find 1. $L^{-1}\left[\frac{1}{(s-4)^3}\right]$ 2. $L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right]$ 3. $L^{-1}\left[\frac{s}{s^2+6s+13}\right]$

4. $L^{-1}\left[\frac{s-2}{s^2+2s+2}\right]$ 5. $L^{-1}\left[\frac{s+1}{s^2+2s}\right]$ 6. $L^{-1}\left[\frac{s+1}{s^2+2s+10}\right]$

7. $L^{-1}\left[\frac{s-2}{s^2+6s-16}\right]$ 8. $L^{-1}\left[\frac{s+1}{s^2+s+1}\right]$ 9. $L^{-1}\left[\frac{c\,s+d}{(s+a)^2+b^2}\right]$

<u>Type – 2</u>

$$\text{If } L[f(t)] = F(s), \text{ then } L\,[t\,f(t)] = -\frac{d}{ds}F(s) = -F'(s)$$

$$\Rightarrow L^{-1}[F'(s)] = -t\,f(t)$$

$$\Rightarrow L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

Problems

1. *Find* $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Sol. *Let* $F'(s) = \frac{s}{(s^2+a^2)^2}$

$$put\ s^2+a^2 = t$$
$$2s\,ds = dt$$
$$s\,ds = dt/2$$

$$\Rightarrow F(s) = \int \frac{s}{(s^2+a^2)^2}\,ds$$

$$= \int \frac{dt/2}{t^2}$$

$$= \frac{1}{2}\int t^{-2}dt$$

$$= \frac{1}{2}.\frac{t^{-1}}{-1}$$

$$= \frac{-1}{2t} = \frac{-1}{2(s^2+a^2)}$$

$$L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -t\,L^{-1}\left[\frac{-1}{2(s^2+a^2)}\right]$$

$$= \frac{t}{2}L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{t\sin at}{2a}$$

2. *Find* $L^{-1}\left[\dfrac{s+2}{(s^2+4s+5)^2}\right]$

Sol. *Let* $F'(s) = \dfrac{s+2}{(s^2+4s+5)^2}$

put $s^2+4s+5 = t$

$(2s+4)ds = dt$

$2(s+2)ds = dt$

$$\Rightarrow F(s) = \int \frac{s+2}{(s^2+4s+5)^2}\,ds$$

$$= \int \frac{dt/2}{t^2}$$

$$= \frac{1}{2}\int t^{-2}dt$$

$$= \frac{1}{2}.\frac{t^{-1}}{-1}$$

$$= \frac{-1}{2t} = \frac{-1}{2(s^2+4s+5)}$$

$$L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

$$\therefore L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = -t\,L^{-1}\left[\frac{-1}{2(s^2+4s+5)}\right]$$

$$= \frac{t}{2}L^{-1}\left[\frac{1}{(s+2)^2+1}\right] = \frac{t}{2}e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{t}{2}e^{-2t}\sin t.$$

Home Work

Find 1. $L^{-1}\left[\dfrac{s}{(s^2-1)^2}\right]$ 2. $L^{-1}\left[\dfrac{s}{(s^2+4)^2}\right]$ 3. $L^{-1}\left[\dfrac{s+3}{(s^2+6s+13)^2}\right]$

4. $L^{-1}\left[\dfrac{2(s+1)}{(s^2+2s+2)^2}\right]$

Type – 3

$$If\ \ L[f(t)] = F(s),\ \ then\ \ L\,[t\,f(t)] = -\frac{d}{ds}F(s)$$

Problems

1. *Find* $L^{-1}\left[\log\left(\frac{1-s^2}{s^2}\right)\right]$

Sol. $L^{-1}\left[\log\left(\frac{1-s^2}{s^2}\right)\right] = f(t)$

$$\Rightarrow L[f(t)] = \log\left(\frac{1-s^2}{s^2}\right)$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}\log\left(\frac{1-s^2}{s^2}\right)$$

$$= -\frac{d}{ds}[\log(1-s^2) - \log s^2]$$

$$= -\left[\frac{-2s}{1-s^2} - \frac{2s}{s^2}\right]$$

$$= \frac{-2s}{s^2-1} + \frac{2}{s}$$

$$\Rightarrow t\,f(t) = -2L^{-1}\left[\frac{s}{s^2-1}\right] + 2L^{-1}\left[\frac{1}{s}\right]$$

$$= -2\cosh t + 2(1)$$

$$(i.e.)\ f(t) = \frac{2}{t}(1-\cosh t)$$

2. *Find* $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

Sol. $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = f(t)$

$$\Rightarrow L[f(t)] = \tan^{-1}\left(\frac{1}{s}\right)$$

$$\therefore\ L[t\,f(t)] = -\frac{d}{ds}\tan^{-1}\left(\frac{1}{s}\right)$$

$$= -\left[\frac{1}{1+(1/s)^2}.\left(\frac{-1}{s^2}\right)\right]$$

$$= \frac{s^2}{s^2+1}.\left(\frac{1}{s^2}\right) = \frac{1}{s^2+1}$$

$$\Rightarrow t\, f(t) = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \sin t$$

$$(i.e.)\ f(t) = \frac{\sin t}{t}$$

Home Work

Find 1. $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$ 2. $L^{-1}\left[\log\left(\frac{1+s}{s}\right)\right]$ 3. $L^{-1}\left[\log\left(\frac{s}{s^2+1}\right)\right]$

4. $L^{-1}\left[\cot^{-1} s\right]$

Type – 4

If $L[\phi(t)] = s\,F(s)$ *and* $f(t)$ *is a function such that* $L[f(t)] = F(s)$ *and* $f(0) = 0$ *then* $\phi(t) = f(t)$.

For, we have $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= s\,F(s) - 0$$

$$= L[\phi(t)]$$

$$\Rightarrow f'(t) = \phi(t)$$

This result can be used to get the inverse transform of certain functions.

$$L^{-1}\left[s\,F(s)\right] = \phi(t)$$

$$= f'(t) = \frac{d}{dt} f(t) = \frac{d}{dt} L^{-1}[F(s)]$$

$(i.e.)\ L^{-1}\left[s\,F(s)\right] = \frac{d}{dt} L^{-1}[F(s)]$ *provided* $L^{-1}[F(s)] = f(t)$ *vanishes at* $t = 0$.

Note $: L^{-1}\left[s^2 F(s)\right] = \frac{d^2}{dt^2} L^{-1}[F(s)]$ *provided* $f(0) = 0, f'(0) = 0$ *when* $f(t) = L^{-1}[F(s)]$.

Problems

1. *Find* $L^{-1}\left[\dfrac{s}{(s+2)^2}\right]$

Sol. $L^{-1}\left[\dfrac{s}{(s+2)^2}\right]=\dfrac{d}{dt}L^{-1}\left[\dfrac{1}{(s+2)^2}\right]$

$$=\frac{d}{dt}\left[e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)\right]$$

$$=\frac{d}{dt}\left[e^{-2t}\,t\right]$$

$$=e^{-2t}.1+t.e^{-2t}(-2)$$

$$=e^{-2t}(1-2t).$$

2. *Find* $L^{-1}\left[\dfrac{s^2}{(s-1)^3}\right]$

Sol. $L^{-1}\left[\dfrac{s^2}{(s-1)^3}\right]=\dfrac{d^2}{dt^2}L^{-1}\left[\dfrac{1}{(s-1)^3}\right]$

$$=\frac{d^2}{dt^2}\left[e^tL^{-1}\left(\frac{1}{s^3}\right)\right]\;=\frac{d^2}{dt^2}\left[e^t\left(\frac{t^2}{2}\right)\right]$$

$$=\frac{1}{2}\frac{d}{dt}[e^t.2t+t^2.e^t]$$

$$=\frac{1}{2}\,[\{e^t.2+2t.e^t\}+\{e^t.2t+t^2.e^t\}]$$

$$=\frac{e^t}{2}[t^2+4t+2]$$

3. Find the inverse transform of $\dfrac{1}{s^{-2}(s^2+1)}$.

Sol. $L^{-1}\left[\dfrac{1}{s^{-2}(s^2+1)}\right]=L^{-1}\left[\dfrac{s^2}{s^2+1}\right]=L^{-1}\left[\dfrac{(s^2+1)-1}{s^2+1}\right]$

$$=L^{-1}(1)-L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$=\delta_0(t)-\sin t$$

$$4.\ Find\ \ L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$$

$$Sol.\ L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]=L^{-1}\left[s.\frac{s}{(s^2+a^2)^2}\right]=\frac{d}{dt}L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] \text{ ----(1)}$$

$$Let\ \ F'(s)=\frac{s}{(s^2+a^2)^2}$$

$$put\ s^2+a^2=t$$

$$2s\,ds=dt$$

$$s\,ds=dt/2$$

$$\Rightarrow F(s)=\int\frac{s}{(s^2+a^2)^2}ds$$

$$=\int\frac{dt/2}{t^2}$$

$$=\frac{1}{2}\int t^{-2}dt$$

$$=\frac{1}{2}.\frac{t^{-1}}{-1}$$

$$=\frac{-1}{2t}=\frac{-1}{2(s^2+a^2)}$$

$$L^{-1}[F'(s)]=-t\,L^{-1}[F(s)]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=-t\,L^{-1}\left[\frac{-1}{2(s^2+a^2)}\right]$$

$$=\frac{t}{2}L^{-1}\left[\frac{1}{s^2+a^2}\right]=\frac{t\sin at}{2a}$$

Equation (1) becomes

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]=\frac{d}{dt}\left[\frac{t\sin at}{2a}\right]$$

$$=\frac{1}{2a}[\sin at.1+t\,(a\cos at)]$$

$$=\frac{\sin at+a\,t\cos at}{2a}$$

Home Work

$$Find\ \ 1.\ L^{-1}\left[\frac{s}{(s+3)^5}\right] \qquad 2.\ L^{-1}\left[\frac{s^2}{(s-1)^4}\right]$$

Type – 5

$$We\ know\ that\ \ L\left[\int_0^t f(t)\,dt\right]=\frac{1}{s}L[f(t)]$$

This result can be used to get the inverse transform of certain functions.

$$\Rightarrow\ L\left[\int_0^t f(t)\,dt\right]=\frac{1}{s}F(s)$$

$$\Rightarrow\ L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t f(t)\,dt$$

$$(i.e.)\ L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t L^{-1}[F(s)]\ dt$$

Problems

1. *Find* $L^{-1}\left[\dfrac{1}{s(s+a)}\right]$

$$Sol.\ \ L^{-1}\left[\frac{1}{s(s+a)}\right]=\int_0^t L^{-1}\left(\frac{1}{s+a}\right)\,dt$$

$$=\int_0^t e^{-at}dt$$

$$=\left[\frac{e^{-at}}{-a}\right]_0^t=\frac{1}{-a}\left[e^{-at}-1\right]=\frac{1-e^{-at}}{a}$$

2. *Find* $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]$

$$Sol.\ \ L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{1}{s}.\frac{s}{(s^2+a^2)^2}\right]=\int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt\ \ ----(1)$$

$$Let\ \ F'(s)=\frac{s}{(s^2+a^2)^2}$$

$$put\ s^2+a^2=t$$
$$2s\,ds=dt$$
$$s\,ds=dt/2$$

$$\Rightarrow F(s)=\int\frac{s}{(s^2+a^2)^2}ds$$

$$=\int\frac{dt/2}{t^2}$$

$$=\frac{1}{2}\int t^{-2}dt$$

$$=\frac{1}{2}.\frac{t^{-1}}{-1}$$

$$=\frac{-1}{2t}=\frac{-1}{2(s^2+a^2)}$$

$$L^{-1}\left[F'(s)\right]=-t\,L^{-1}[F(s)]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=-t\,L^{-1}\left[\frac{-1}{2(s^2+a^2)}\right]$$

$$=\frac{t}{2}L^{-1}\left[\frac{1}{s^2+a^2}\right]\;=\frac{t\sin at}{2a}$$

Equation (1) becomes

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=\int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt$$

$$=\int_0^t \frac{t\sin a\,t}{2a}dt$$

$$=\frac{1}{2a}\left[t\left(\frac{-\cos at}{a}\right)-(1)\left(\frac{-\sin at}{a^2}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[\left\{\frac{-t\cos at}{a}+\frac{\sin at}{a^2}\right\}-\{0+0\}\right]$$

$$=\frac{\sin at-a\,t\cos at}{2a^3}.$$

Home Work

Find 1. $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$ 2. $L^{-1}\left[\frac{1}{s\,(s-a)}\right]$ 3. $L^{-1}\left[\frac{1}{s\,(s+2)^3}\right]$

Type – 6

Method of Partial fractions

To resolve into partial fractions, degree of numerator should be less than the degree of denominator. [deg. of Nr. < deg. of Dr.]

1. $\frac{1}{(x+a)(x+b)}=\frac{A}{x+a}+\frac{B}{x+b}$
2. $\frac{1}{(x+a)(x+b)(x+c)}=\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c}$
3. $\frac{1}{(x+a)(x+b)(x+c)(x+d)}=\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c}+\frac{D}{x+d}$
4. $\frac{x}{(x+a)^2(x+b)}=\frac{A}{x+a}+\frac{B}{(x+a)^2}+\frac{C}{x+b}$
5. $\frac{x^3+x+2}{(x+a)(x+b)^3}=\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{(x+b)^2}+\frac{D}{(x+b)^3}$
6. $\frac{x^2+3}{(x^2+a)(x+b)}=\frac{Ax+B}{x^2+a}+\frac{C}{x+b}$
7. $\frac{x^2(or)cons\tan t}{(x^2+a)(x^2+b)}=\frac{A}{x^2+a}+\frac{B}{x^2+b}$
8. $\frac{x^3(or)x}{(x^2+a)(x^2+b)}=\frac{Ax+B}{x^2+a}+\frac{Cx+D}{x^2+b}$
9. $\frac{x^3+1}{(x+a)(x+b)^2}=A+\frac{B}{x+a}+\frac{C}{x+b}+\frac{D}{(x+b)^2}$
10. $\frac{x^3+x-1}{(x+a)(x+b)}=Ax+B+\frac{C}{x+a}+\frac{D}{x+b}$

Problems

1. $Find\ \ L^{-1}\left[\dfrac{s+2}{(s+1)(s+4)}\right]$

$$Sol.\quad \frac{s+2}{(s+1)(s+4)}=\frac{A}{s+1}+\frac{B}{s+4}$$

$$s+2=A\,(s+4)+B(s+1)$$

$$Put\ \ s=-1,\quad -1+2=A(-1+4)+B(0)$$

$$1=3A$$

$$A=\frac{1}{3}$$

$$Put\ \ s=-4,\quad -4+2=A(0)+B(-4+1)$$

$$-2=-3A$$

$$B=\frac{2}{3}$$

$$\frac{s+2}{(s+1)(s+4)}=\frac{1/3}{s+1}+\frac{2/3}{s+4}$$

$$L^{-1}\left[\frac{s+2}{(s+1)(s+4)}\right]=\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{2}{3}L^{-1}\left[\frac{1}{s+4}\right]$$

$$=\frac{1}{3}e^{-t}+\frac{2}{3}e^{-4t}$$

2. $Find\ \ L^{-1}\left[\dfrac{1}{s\,(s+1)(s+2)}\right]$

$$Sol.\quad \frac{1}{s\,(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

$$1=A\,(s+1)(s+2)+B\,s\,(s+2)+C\,s\,(s+1)$$

$$Put\ \ s=0,\qquad 1=A(1)(2)+B(0)+C(0)$$

$$1=2A$$

$$A=\frac{1}{2}$$

$$Put\ \ s=-1,\qquad 1=A(0)+B(-1)(1)+C(0)$$

$$1=-B$$

$$B=-1$$

$$Put\ s=-2, \qquad 1 = A(0)+B(0)+C(-2)(-1)$$

$$1 = 2C$$

$$C=\frac{1}{2}$$

$$\frac{1}{s(s+1)(s+2)}=\frac{1/2}{s}-\frac{1}{s+1}+\frac{1/2}{s+2}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}$$

3. $Find\ \ L^{-1}\left[\frac{s+2}{(s+3)(s^2+4)}\right]$

$Sol.$ $\frac{s+2}{(s+3)(s^2+4)}=\frac{A}{s+3}+\frac{Bs+C}{s^2+4}$

$$s + 2 = A (s^2 + 4) + (Bs + C)(s + 3)$$

put s = –3, $\quad -3 + 2 = A(9 + 4) + 0$

$$-1 = A(13) \Rightarrow A = -\frac{1}{13}$$

Coeff. of s^2, $\quad 0 = A + B$

$$B = -A \Rightarrow B = \frac{1}{13}$$

Coeff. of s, $\quad 1 = 3B + C$

$$C = 1 - 3B$$

$$C = 1 - 3(\frac{1}{13}) \Rightarrow C = \frac{10}{13}$$

$$\frac{s+2}{(s+3)(s^2+4)}=\frac{-1/13}{s+3}+\frac{(1/13)s+10/13}{s^2+4}$$

$$L^{-1}\left[\frac{s+2}{(s+3)(s^2+4)}\right]=-\frac{1}{13}L^{-1}\left(\frac{1}{s+3}\right)+L^{-1}\left[\frac{\frac{1}{13}s+\frac{10}{13}}{s^2+4}\right]$$

$$=-\frac{1}{13}e^{-3t}+\frac{1}{13}L^{-1}\left(\frac{s}{s^2+4}\right)+\frac{10}{13}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$=\frac{1}{13}\left[-e^{-3t}+\cos 2t+10.\frac{\sin 2t}{2}\right]$$

$$=\frac{1}{13}\left[-e^{-3t}+\cos 2t+5\sin 2t\right].$$

4. *Find* $L^{-1}\left[\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right]$

Sol. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}=\dfrac{A}{s-1}+\dfrac{Bs+C}{s^2+2s+5}$

$5s + 3 = A(s^2 + 2s + 5) + (Bs + C)(s - 1)$

put s = 1, $5 + 3 = A(1 + 2 + 5) + 0$

$8 = A(8) \Rightarrow A = 1$

Coeff. of s^2, $0 = A + B$

$B = -A \Rightarrow B = -1$

Coeff. of s, $5 = 2A - B + C$

$5 = 2(1) - (-1) + C$

$C = 5 - 3 \Rightarrow C = 2$

$$\frac{5s+3}{(s-1)(s^2+2s+5)}=\frac{1}{s-1}+\frac{(-1)s+2}{s^2+2s+5}$$

$$L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]=L^{-1}\left(\frac{1}{s-1}\right)+L^{-1}\left[\frac{-s+2}{s^2+2s+5}\right]$$

$$=e^t-L^{-1}\left[\frac{s-2}{(s+1)^2+4}\right]$$

$$=e^t-L^{-1}\left[\frac{(s+1)-3}{(s+1)^2+4}\right]$$

$$=e^t-L^{-1}\left[\frac{s+1}{(s+1)^2+4}\right]+3L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$=e^t-e^{-t}L^{-1}\left[\frac{s}{s^2+4}\right]+3e^{-t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$=e^t-e^{-t}\cos 2t+3e^{-t}\frac{\sin 2t}{2}$$

5. *Find* $L^{-1}\left[\dfrac{1+2s}{(s+2)^2(s-1)^2}\right]$

Sol. $\dfrac{1+2s}{(s+2)^2(s-1)^2}=\dfrac{A}{s+2}+\dfrac{B}{(s+2)^2}+\dfrac{C}{s-1}+\dfrac{D}{(s-1)^2}$

$$1+2s=A(s+2)(s-1)^2+B(s-1)^2+C(s-1)(s+2)^2+D(s+2)^2$$

Put s = 1, $1 + 2 = A(0) + B(0) + C(0) + D(3)^2$

$3 = 9D \Rightarrow D = \dfrac{1}{3}$

Put s = – 2, $1- 4 = A(0) + B(-3)^2 + C(0) +D(0)$

$-3 = 9B \Rightarrow B = -\dfrac{1}{3}$

Coeff. of s^3, $0 = A + C$ --------------(1)

Coeff. of s^2, $0 = 0 + B + 3C + D$

$$0 = -\frac{1}{3} + 3C + \frac{1}{3}$$

$$0 = 3C \Rightarrow C = 0$$

From (1), we have A = 0

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{0}{s+2} + \frac{-1/3}{(s+2)^2} + \frac{0}{s-1} + \frac{1/3}{(s-1)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= -\frac{1}{3}\, e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] + \frac{1}{3}\, e^{t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= -\frac{1}{3}\, e^{-2t}\, t + \frac{1}{3}\, e^{t}\, t$$

$$= \frac{t}{3}(e^{t} - e^{-2t})$$

$A(s+2)(s-1)^2$

$A(s+2)(s^2-2s+1)$

$A(s^3+0s^2-3s+2)$

$C(s-1)(s+2)^2$

$C(s-1)(s^2+4s+4)$

$C(s^3+3s^2+0s-4)$

6. *Find* $L^{-1}\left[\frac{s^2+3}{(s^2-4)(s^2+16)}\right]$

Sol. $\frac{s^2+3}{(s^2-4)(s^2+16)} = \frac{A}{s^2-4} + \frac{B}{s^2+16}$

$$s^2+3 = A(s^2+16) + B(s^2-4)$$

Put $s^2 = 4$, $7 = A(20) + B(0) \Rightarrow A = \frac{7}{20}$

Put $s^2 = -16$, $-13 = A(0) + B(-20) \Rightarrow B = \frac{13}{20}$

$$\frac{s^2+3}{(s^2-4)(s^2+16)} = \frac{7}{20}\frac{1}{s^2-4} + \frac{13}{20}\frac{1}{s^2+16}$$

$$L^{-1}\left[\frac{s^2+3}{(s^2-4)(s^2+16)}\right] = \frac{7}{20}L^{-1}\left(\frac{1}{s^2-4}\right) + \frac{13}{20}L^{-1}\left(\frac{1}{s^2+16}\right)$$

$$= \frac{7}{20}.\frac{\sinh 2t}{2} + \frac{13}{20}.\frac{\sin 4t}{4}$$

$$= \frac{7}{40}\sinh 2t + \frac{13}{80}\sin 4t$$

7. *Find* $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}\right]$

Sol. $\dfrac{1}{(s^2+1)(s^2+4)(s^2+9)} = \dfrac{A}{s^2+1} + \dfrac{B}{s^2+4} + \dfrac{C}{s^2+9}$

$$1 = A(s^2+4)(s^2+9) + B(s^2+1)(s^2+9) + C(s^2+1)(s^2+4)$$

$$Put\ s^2 = -1, \quad 1 = A(3)(8) \Rightarrow A = \frac{1}{24}$$

$$Put\ s^2 = -4, \quad 1 = B(-3)(5) \Rightarrow B = -\frac{1}{15}$$

$$Put\ s^2 = -9, \quad 1 = C(-8)(-5) \Rightarrow C = \frac{1}{40}$$

$$\frac{1}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1/24}{s^2+1} + \frac{-1/15}{s^2+4} + \frac{1/40}{s^2+9}$$

$$\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1}{24}\frac{s}{s^2+1} - \frac{1}{15}\frac{s}{s^2+4} + \frac{1}{40}\frac{s}{s^2+9}$$

$$L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right] = \frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{24}\cos t - \frac{1}{15}\cos 2t + \frac{1}{40}\cos 3t.$$

Home Work

Find 1. $L^{-1}\left[\dfrac{1}{s^2(s+1)}\right]$ 2. $L^{-1}\left[\dfrac{1}{(s+1)(s^2+2s+2)}\right]$ 3. $L^{-1}\left[\dfrac{1}{(s+1)(s^2+1)}\right]$

4. $L^{-1}\left[\dfrac{s^2-s+2}{s(s-3)(s+2)}\right]$ 5. $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+4)}\right]$ 6. $L^{-1}\left[\dfrac{s+4}{s(s-1)(s^2+4)}\right]$

7. $L^{-1}\left[\dfrac{1}{s(s^2-2s+5)}\right]$ 8. $L^{-1}\left[\dfrac{1}{(s-2)(s^2+1)}\right]$

Type – 7

By second shifting property, we have

$$L\,[f(t-a)u_a(t)] = e^{-as}F(s)$$

$$L\,[G(t)] = e^{-as}F(s) \;\; where \;\; G(t) = f(t-a)\,u_a(t)$$

$$\Rightarrow L^{-1}\left[e^{-as}F(s)\right] = G(t)$$

$$= f(t-a),\; t>a$$

$$\Rightarrow L^{-1}\left[e^{-as}F(s)\right] = L^{-1}\left[F(s)\right]_{t\to t-a}$$

Unit Step function is

$$u_a(t) = \begin{cases} 1 & when \;\; t>a \\ 0 & when \;\; t<a \end{cases}$$

$$G(t) = f(t-a)u_a(t)$$

$$= \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$$

Problems

1\. $Find \;\; L^{-1}\left[\dfrac{e^{-2s}}{s^2+4s+13}\right]$

$$Sol. \;\; L^{-1}\left[\frac{e^{-2s}}{s^2+4s+13}\right] = L^{-1}\left[\frac{1}{s^2+4s+13}\right]_{t\to t-2}$$

$$Now, \;\; L^{-1}\left[\frac{1}{s^2+4s+13}\right] = L^{-1}\left[\frac{1}{(s+2)^2+9}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-2t}\,\frac{\sin 3t}{3}$$

$$\therefore \;\; L^{-1}\left[\frac{e^{-2s}}{s^2+4s+13}\right] = \left[\frac{e^{-2t}\sin 3t}{3}\right]_{t\to t-2}$$

$$= \frac{e^{-2(t-2)}\sin 3(t-2)}{3},\; t>2$$

2\. $If \;\; L[f(t)] = \dfrac{e^{-s}}{(s+1)(s+3)}, \;\; find \;\; f(t).$

$$Sol. \;\; Given \; L[f(t)] = \frac{e^{-s}}{(s+1)(s+3)} \;\Rightarrow\; f(t) = L^{-1}\left[\frac{e^{-s}}{(s+1)(s+3)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]_{t\to t-1}$$

$$Now, \;\; \frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$1 = A(s+3) + B(s+1)$$

$$Put \;\; s=-1, \qquad 1 = A(-1+3) + B(0)$$

$$1 = 2A \;\Rightarrow\; A = \frac{1}{2}$$

$$Put \;\; s=-3, \qquad 1 = A(0) + B(-3+1)$$

$$1 = -2B \;\Rightarrow\; B = -\frac{1}{2}$$

$$\frac{1}{(s+1)(s+3)}=\frac{1/2}{s+1}-\frac{1/2}{s+3}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+3)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right]-\frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}$$

$$\therefore\ f(t)=L^{-1}\left[\frac{e^{-s}}{(s+1)(s+3)}\right]=\left[\frac{1}{2}e^{-t}-\frac{1}{2}e^{-2t}\right]_{t\to t-1}$$

$$=\frac{1}{2}e^{-(t-1)}-\frac{1}{2}e^{-2(t-1)},\ t>1$$

3. *Find* $L^{-1}\left[\frac{e^{-3s}}{(s-2)^4}\right]$

Sol. $L^{-1}\left[\frac{e^{-3s}}{(s-2)^4}\right]=L^{-1}\left[\frac{1}{(s-2)^4}\right]_{t\to t-3}$

Now, $L^{-1}\left[\frac{1}{(s-2)^4}\right]=e^{2t}L^{-1}\left[\frac{1}{s^4}\right]=e^{2t}\frac{t^3}{3!}=\frac{t^3\,e^{2t}}{6}$

$$\therefore\ L^{-1}\left[\frac{e^{-3s}}{(s-2)^4}\right]=\left[\frac{t^3e^{2t}}{6}\right]_{t\to t-3}$$

$$=\frac{(t-3)^3e^{2(t-3)}}{6},\ t>3$$

Home Work

1. *Find* $L^{-1}\left[\frac{e^{-2s}}{s(s^2+9)}\right]$

2. *If* $L[f(t)]=e^{-2s}\tan^{-1}s$, *find* $f(t)$.

Convolution of two functions

If f(t) and g(t) are given functions then the convolution of f(t) and g(t) is defined as $\int_0^t f(u)g(t-u)du$. It is denoted by $f(t)*g(t)$

$$(i.e.)\quad f(t)*g(t)=\int_0^t f(u)g(t-u)du$$

Convolution theorem on Laplace transform.

If f(t) and g(t) are functions defined for t ≥ 0, then L[$f(t)*g(t)$] = F(s).G(s)

Note: 1) Using the above theorem, we get

$$L^{-1}[F(s)G(s)]=f(t)*g(t)$$
$$=L^{-1}[F(s)]*L^{-1}[G(s)]$$

2) $f(t)*g(t)=g(t)*f(t)$

Problems

1. *Find* $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$ $u\sin g$ *convolution theorem.*

Sol. $$L^{-1}\left[\frac{1}{s(s^2+1)}\right]=L^{-1}\left(\frac{1}{s}\right)*L^{-1}\left(\frac{1}{s^2+1}\right)$$
$$=1*\sin t$$
$$=\sin t*1$$
$$=\int_0^t \sin u.1du \quad =\left[-\cos u\right]_0^t$$
$$=-\left[\cos t-\cos 0\right]$$
$$=-[\cos t-1]$$
$$=1-\cos t$$

2. *Find* $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ $u\sin g$ *convolution theorem.*

Sol. $$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right]$$
$$=L^{-1}\left(\frac{s}{s^2+a^2}\right)*L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a}\int_0^t \cos au . \sin a(t-u)\, du$$

2sinAcosB = sin(A + B) + sin(A – B)

$$= \frac{1}{2a}\int_0^t 2\sin a(t-u)\cos au \;\, du$$

$$= \frac{1}{2a}\int_0^t [\sin(at-au+au)+\sin(at-au-au)]\, du$$

$$= \frac{1}{2a}\int_0^t [\sin(at)+\sin(at-2au)]\, du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{\cos(at-2au)}{-2a}\right]_0^t$$

[cos(at – 2at) = cos(– at)
= cosat]

$$= \frac{1}{2a}\left[\left\{t\sin at + \frac{\cos at}{2a}\right\} - \left\{0 + \frac{\cos at}{2a}\right\}\right]$$

$$= \frac{t\sin at}{2a}$$

3. *Find* $L^{-1}\left[\frac{1}{(s^2+4)^2}\right] u\sin g$ *convolution theorem.*

Sol. $L^{-1}\left[\frac{1}{(s^2+4)^2}\right] = L^{-1}\left(\frac{1}{s^2+4}\right) * L^{-1}\left(\frac{1}{s^2+4}\right)$

$$= \frac{\sin 2t}{2} * \frac{\sin 2t}{2}$$

$$= \frac{1}{4}\int_0^t \sin 2u . \sin 2(t-u)\, du$$

$$= \frac{1}{8}\int_0^t 2\sin 2(t-u)\sin 2u \;\, du$$

2sinAsinB = cos(A – B) – cos(A + B)

$$= \frac{1}{8}\int_0^t [\cos(2t-2u-2u)-\cos(2t-2u+2u)]\, du$$

$$= \frac{1}{8}\int_0^t [\cos(2t-4u)-\cos 2t]\, du$$

$$=\frac{1}{8}\left[\frac{\sin(2t-4u)}{-4}-u\cos 2t\right]_0^t$$

[sin(2t – 4t) = sin(–2t)
= – sin2t]

$$=\frac{1}{8}\left[\left\{\frac{-\sin 2t}{-4}-t\cos 2t\right\}-\left\{\frac{\sin 2t}{-4}-0\right\}\right]$$

$$=\frac{1}{8}\left[\frac{2\sin 2t}{4}-t\cos 2t\right]$$

$$=\frac{1}{8}\left[\frac{\sin 2t}{2}-t\cos 2t\right]$$

4. Using convolution theorem find the inverse Laplace transform of the following

$$(a)\ \frac{s+2}{(s^2+4s+13)^2} \qquad (b)\ \frac{2}{(s+1)(s^2+4)}$$

Sol.

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]=L^{-1}\left(\frac{s+2}{(s+2)^2+9}\right)*L^{-1}\left(\frac{1}{(s+2)^2+9}\right)$$

$$=e^{-2t}\cos 3t * e^{-2t}\frac{\sin 3t}{3}$$

$$=\frac{1}{3}\int_0^t e^{-2u}\cos 3u\ e^{-2(t-u)}\sin 3(t-u)\,du$$

$$=\frac{1}{3}\int_0^t e^{-2u-2t+2u}\cos 3u\ \sin 3(t-u)\,du$$

$$=\frac{1}{6}\int_0^t e^{-2t}\,[2\sin(3t-3u)\ \cos 3u]\ du$$

$$=\frac{e^{-2t}}{6}\int_0^t [\sin(3t-3u+3u)+\sin(3t-3u-3u)]du$$

$$=\frac{e^{-2t}}{6}\int_0^t [\sin 3t+\sin(3t-6u)]du$$

$$=\frac{e^{-2t}}{6}\left[u\sin 3t-\frac{\cos(3t-6u)}{-6}\right]_0^t$$

$$=\frac{e^{-2t}}{6}\left[\left\{t\sin 3t+\frac{\cos 3t}{6}\right\}-\left\{0+\frac{\cos 3t}{6}\right\}\right]$$

$$=\frac{te^{-2t}\sin 3t}{6}$$

$$(b)\ L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right] = L^{-1}\left(\frac{1}{s+1}\right) * L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= e^{-t} * \sin 2t$$

$$= \int_0^t e^{-u} \sin 2(t-u)\, du$$

$$\boxed{\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)}$$

$$= \left[\frac{e^{-u}}{1+4}\{-\sin(2t-2u) + 2\cos(2t-2u)\}\right]_0^t$$

$$= \left\{\frac{e^{-t}}{5}(0+2)\right\} - \left\{\frac{1}{5}(-\sin 2t + 2\cos 2t)\right\}$$

$$= \frac{1}{5}\left[2e^{-t} + \sin 2t - 2\cos 2t\right]$$

Home Work

1. Using convolution theorem find the inverse Laplace transform of the following.

$$(a)\ \frac{1}{s^2(s^2+25)} \quad (b)\ \frac{1}{s^3(s+5)} \quad (c)\ \frac{1}{s(s^2-a^2)} \quad (d)\ \frac{4}{(s^2+2s+5)^2}$$

2. Using convolution theorem find $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$

Solving Ordinary Differential Equations using Laplace Transform

1. Use Laplace transform method to solve

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=4, with\ y=2,\ \frac{dy}{dt}=3\ at\ t=0.$$

Sol. Let $y=f(t)$. Then the given equation becomes

$$f''(t)-3f'(t)+2f(t)=4$$

Taking Laplace transform on both sides, we get

$$L[f''(t)]-3L[f'(t)]+2L[f(t)]=L(4)$$

$$\left\{s^2L[f(t)]-sf(0)-f'(0)\right\}-3\left\{sL[f(t)]-f(0)\right\}+2L[f(t)]=\frac{4}{s}$$

$$Given\quad y(t)=2,\ y'(t)=3\ \ when\ t=0$$

$$y(0)=2,\ y'(0)=3$$

$$(i.e.)\ f(0)=2,\ f'(0)=3$$

$$\left\{s^2L[f(t)]-2s-3\right\}-3\left\{sL[f(t)]-2\right\}+2L[f(t)]=\frac{4}{s}$$

$$(s^2-3s+2)L[f(t)]=\frac{4}{s}+2s+3-6$$

$$(s-1)(s-2)L[f(t)]=\frac{4}{s}+2s-3=\frac{4+2s^2-3s}{s}$$

$$L[f(t)]=\frac{2s^2-3s+4}{s(s-1)(s-2)}$$

$$f(t)=L^{-1}\left[\frac{2s^2-3s+4}{s(s-1)(s-2)}\right]$$

$$\frac{2s^2-3s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}$$

$$2s^2-3s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1)$$

$$Put\ \ s=0,\qquad 4=A(-1)(-2)+B(0)+C(0)$$

$$4=2A$$

$$A=2$$

$$Put\ \ s=1,\qquad 2-3+4=A(0)+B(1)(-1)+C(0)$$

$$3=-B$$

$$B=-3$$

$$Put\ \ s=2,\qquad 8-6+4=A(0)+B(0)+C(2)(1)$$

$$6=2C$$

$$C=3$$

$$\frac{2s^2-3s+4}{s(s-1)(s-2)}=\frac{2}{s}-\frac{3}{s-1}+\frac{3}{s-2}$$

$$f(t)=L^{-1}\left[\frac{2s^2-3s+4}{s(s-1)(s-2)}\right]=2L^{-1}\left[\frac{1}{s}\right]-3L^{-1}\left[\frac{1}{s-1}\right]+3L^{-1}\left[\frac{1}{s-2}\right]$$

$$(i.e.)\ y=2-3e^t+3e^{2t}$$

2. Use Laplace transform method to solve

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+13x=2e^{-t}\quad given\ x(0)=0,\ x'(0)=-1$$

Sol. Let $x=f(t)$. Then the given equation becomes

$$f''(t)+4f'(t)+13f(t)=2e^{-t}$$

Taking Laplace transform on both sides, we get

$$L[f''(t)]+4L[f'(t)]+13L[f(t)]=2L(e^{-t})$$

$$\left\{s^2L[f(t)]-sf(0)-f'(0)\right\}+4\left\{sL[f(t)]-f(0)\right\}+13L[f(t)]=\frac{2}{s+1}$$

$$Given\quad x(0)=0,\ x'(0)=-1$$

$$(i.e.)\ f(0)=0,\ f'(0)=-1$$

$$\left\{s^2L[f(t)]-0+1\right\}+4\left\{sL[f(t)]-0\right\}+13L[f(t)]=\frac{2}{s+1}$$

$$(s^2+4s+13)L[f(t)]=\frac{2}{s+1}-1=\frac{2-(s+1)}{s+1}$$

$$L[f(t)]=\frac{1-s}{(s+1)(s^2+4s+13)}$$

$$f(t)=L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$$

$$\frac{1-s}{(s+1)(s^2+4s+13)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4s+13}$$

$1 - s = A(s^2 + 4s + 13) + (Bs + C)(s + 1)$

put s = –1, $2 = A(1 - 4 + 13) + 0$

$$2 = A(10) \Rightarrow A = \frac{1}{5}$$

Coeff. of s^2, $0 = A + B$

$$B = -A \Rightarrow B = -\frac{1}{5}$$

Coeff. of s, $-1 = 4A + B + C$

$$-1=\frac{4}{5}-\frac{1}{5}+C$$

$$-1=\frac{3}{5}+C$$

$$C=-1-\frac{3}{5}=-\frac{8}{5}$$

$$\frac{1-s}{(s+1)(s^2+4s+13)}=\frac{1/5}{s+1}+\frac{(-1/5)s-8/5}{s^2+4s+13}$$

$$f(t)=L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]=\frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)-\frac{1}{5}L^{-1}\left[\frac{s+8}{(s+2)^2+9}\right]$$

$$=\frac{1}{5}\,e^{-t}-\frac{1}{5}L^{-1}\left[\frac{(s+2)+6}{(s+2)^2+9}\right]$$

$$=\frac{1}{5}\,e^{-t}-\frac{1}{5}L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right]-\frac{6}{5}L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$=\frac{1}{5}\,e^{-t}-\frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right]-\frac{6}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$=\frac{1}{5}\,e^{-t}-\frac{1}{5}e^{-2t}\cos 3t-\frac{6}{5}e^{-2t}\,\frac{\sin 3t}{3}$$

$$(i.e.)\;\; x=\frac{1}{5}\,e^{-t}-\frac{1}{5}e^{-2t}\cos 3t-\frac{2}{5}e^{-2t}\sin 3t$$

3. Use Laplace transform method to solve

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\,,with\;\; y=2,\;\frac{dy}{dx}=-1\;at\;\; x=0.$$

Sol. Let $y=f(x)$. Then the given equation becomes

$$f''(x)-2f'(x)+f(x)=e^x$$

Taking Laplace transform on both sides, we get

$$L[f''(x)]-2L[f'(x)]+L[f(x)]=L(e^x)$$

$$\left\{s^2L[f(x)]-sf(0)-f'(0)\right\}-2\left\{sL[f(x)]-f(0)\right\}+L[f(x)]=\frac{1}{s-1}$$

Given $y(0)=2,\; y'(0)=-1$

(*i.e.*) $f(0)=2,\; f'(0)=-1$

$$\left\{s^2L[f(x)]-2s+1\right\}-2\left\{sL[f(x)]-2\right\}+L[f(x)]=\frac{1}{s-1}$$

$$(s^2-2s+1)L[f(x)]=\frac{1}{s-1}+2s-5$$

$$(s-1)^2 L[f(x)] = \frac{1}{s-1} + 2s - 5$$

$$L[f(x)] = \frac{1}{(s-1)^3} + \frac{2s-5}{(s-1)^2}$$

$$f(x) = L^{-1}\left[\frac{1}{(s-1)^3}\right] + L^{-1}\left[\frac{2s-5}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^3}\right] + L^{-1}\left[\frac{2(s-1)-3}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s-1)^3}\right] + L^{-1}\left[\frac{2}{s-1}\right] - 3L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^x L^{-1}\left[\frac{1}{s^3}\right] + 2e^x - 3e^x L^{-1}\left[\frac{1}{s^2}\right]$$

$$f(x) = e^x \frac{x^2}{2} + 2e^x - 3e^x . x$$

$$(i.e.)\ y = \frac{x^2 e^x}{2} + 2e^x - 3xe^x$$

4. Using Laplace transform solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that y = 4, $\dot{y} = -2$ when t = 0.

Sol. Let $y = f(t)$. Then the given equation becomes

$$f''(t) + f'(t) = t^2 + 2t$$

Taking Laplace transform on both sides, we get

$$L[f''(t)] + L[f'(t)] = L(t^2) + 2L(t)$$

$$\left\{s^2 L[f(t)] - sf(0) - f'(0)\right\} + \left\{sL[f(t)] - f(0)\right\} = \frac{2}{s^3} + \frac{2}{s^2}$$

$$Given\quad y(0) = 4,\ y'(0) = -2$$

$$(i.e.)\ f(0) = 4,\ f'(0) = -2$$

$$\left\{s^2 L[f(t)] - 4s + 2\right\} + \left\{sL[f(t)] - 4\right\} = \frac{2}{s^3} + \frac{2}{s^2}$$

$$(s^2 + s)L[f(t)] = \frac{2+2s}{s^3} + 4s + 2$$

$$s(s+1)L[f(t)]=\frac{2(s+1)}{s^3}+4s+2$$

$$L[f(t)]=\frac{2}{s^4}+\frac{4s+2}{s(s+1)}$$

$$f(t)=2L^{-1}\left[\frac{1}{s^4}\right]+L^{-1}\left[\frac{4s+2}{s(s+1)}\right]$$

$$=2\frac{t^3}{3!}+L^{-1}\left[\frac{4s+2}{s(s+1)}\right]\text{-------}(1)$$

$$\frac{4s+2}{s(s+1)}=\frac{A}{s}+\frac{B}{s+1}$$

$$4s+2=A(s+1)+B(s)$$

$$Put\ s=0,\quad 2=A(1)+0\Rightarrow A=2$$

$$Put\ s=-1,\ -4+2=0+B(-1)$$

$$-2=-B\Rightarrow B=2$$

$$\frac{4s+2}{s(s+1)}=\frac{2}{s}+\frac{2}{s+1}$$

$$L^{-1}\left[\frac{4s+2}{s(s+1)}\right]=2L^{-1}\left[\frac{1}{s}\right]+2L^{-1}\left[\frac{1}{s+1}\right]$$

$$=2+2e^{-t}$$

Equation (1) becomes

$$f(t)=2.\frac{t^3}{6}+2+2e^{-t}$$

$$(i.e.)\ y=\frac{t^3}{3}+2+2e^{-t}.$$

5. Solve using Laplace transform $\frac{dy}{dt}+3y+2\int_0^t ydt=t\ \ given\ \ y(0)=1$

Sol. Let $y=f(t)$. Then the given equation becomes

$$f'(t)+3f(t)+2\int_0^t f(t)dt=t$$

Taking Laplace transform on both sides, we get

$$L[f'(t)]+3L[f(t)]+2L\left[\int_0^t f(t)dt\right]=L[t]$$

$$\{sL[f(t)]-f(0)\}+3L[f(t)]+2\frac{1}{s}L[f(t)]=\frac{1}{s^2}$$

$$Given \quad y(0)=1 \qquad (i.e.) \quad f(0)=1$$

$$\{sL[f(t)]-1\}+3L[f(t)]+\frac{2}{s}L[f(t)]=\frac{1}{s^2}$$

$$\left(s+3+\frac{2}{s}\right)L[f(t)]=\frac{1}{s^2}+1$$

$$\left(\frac{s^2+3s+2}{s}\right)L[f(t)]=\frac{1}{s^2}+1$$

$$\left(\frac{(s+1)(s+2)}{s}\right)L[f(t)]=\frac{1+s^2}{s^2}$$

$$L[f(t)]=\frac{s^2+1}{s(s+1)(s+2)}$$

$$f(t)=L^{-1}\left[\frac{s^2+1}{s(s+1)(s+2)}\right]$$

$$\frac{s^2+1}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

$$s^2+1=A(s+1)(s+2)+B(s)(s+2)+C(s)(s+1)$$

Put s = 0, 0 + 1 = A(1)(2) + 0 + 0

$$A = \frac{1}{2}$$

Put s = –1, 1 + 1 = 0 + B(–1) + 0

B = – 2

Put s = – 2, 4 + 1 = 0 + 0 + C(–2)(–1)

$$2C = 5 \Rightarrow C=\frac{5}{2}$$

$$\frac{s^2+1}{s(s+1)(s+2)}=\frac{1/2}{s}-\frac{2}{s+1}+\frac{5/2}{s+2}$$

$$f(t)=L^{-1}\left[\frac{s^2+1}{s\,(s+1)(s+2)}\right]=\frac{1}{2}L^{-1}\left(\frac{1}{s}\right)-2L^{-1}\left[\frac{1}{s+1}\right]+\frac{5}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$(i.e.)\, y=\frac{1}{2}-2e^{-t}+\frac{5}{2}e^{-2t}$$

6. Solve using Laplace transform $\frac{dy}{dt} + 2y + \int_0^t y\,dt = 2\cos t \ \ given \ \ y(0) = 1$

Sol. Let $y = f(t)$. Then the given equation becomes

$$f'(t) + 2f(t) + \int_0^t f(t)\,dt = 2\cos t$$

Taking Laplace transform on both sides, we get

$$L[f'(t)] + 2L[f(t)] + L\left[\int_0^t f(t)dt\right] = 2L[\cos t]$$

$$\{sL[f(t)] - f(0)\} + 2L[f(t)] + \frac{1}{s}L[f(t)] = \frac{2s}{s^2+1}$$

$$Given \quad y(0) = 1 \qquad (i.e.)\ \ f(0) = 1$$

$$\{sL[f(t)] - 1\} + 2L[f(t)] + \frac{1}{s}L[f(t)] = \frac{2s}{s^2+1}$$

$$\left(s + 2 + \frac{1}{s}\right)L[f(t)] = \frac{2s}{s^2+1} + 1$$

$$\left(\frac{s^2+2s+1}{s}\right)L[f(t)] = \frac{2s+s^2+1}{s^2+1}$$

$$L[f(t)] = \frac{s}{s^2+1}$$

$$f(t) = L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$(i.e.)\ \ y = \cos t$$

Home Work

Using Laplace transform solve the following:

1. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = t\,e^{-t}, \ y(0) = 0, \ y'(0) = -1$
2. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5 \ given \ y(0) = 0, \ y'(0) = 2$
3. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 4e^{-t} \ \ given \ \ y = \frac{dy}{dt} = 0 \ \ when \ t = 0.$
4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 2 \ \ given \ y(0) = 0, \ y'(0) = 1$
5. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}, \ y(0) = 0, \ y'(0) = 0.$

Answers

Page No. 7

1. $\frac{s^2+18}{s(s^2+36)}$ 2. $\frac{s(s^2+112)}{(s^2+16)(s^2+144)}$ 3. $\frac{s^3+29s}{(s^2+9)(s^2+49)}$ 4. $\frac{4s}{(s^2+1)(s^2+9)}$

5. $\frac{s^2-8}{(s^2+4)(s^2+16)}$ 6. $\frac{1-e^{-4(s+1)}}{s+1}$ 7. $\frac{1+e^{-\pi s}}{s^2+1}$

Page No. 11

1. $\frac{1}{s^2-6s+10}$ 2. $\frac{s+2}{s^2+4s-5}$

3. $\frac{3}{2(s^2-9)}+\frac{3s^2-39}{2(s^4-10s^2+169)}$ 4. $\frac{3(s+4)^2+15}{(s^2+8s+17)(s^2+8s+41)}$

Page No. 18

1. $\frac{2as}{(s^2+a^2)^2}$ 2. $\frac{s^2-4}{(s^2+4)^2}$ 3. $\frac{30s^2-250}{(s^2+25)^3}$ 4. $\frac{s^4+2s^2+8}{s^2(s^2+4)^2}$

5. $\frac{s(s+2)}{(s^2+2s+2)^2}$ 6. $\frac{6s+12}{(s^2+4s+13)^2}$ 7. $\frac{4(3s^2+18s+23)}{(s^2+6s+13)^3}$ 8. $\frac{2s^3-12s^2+32}{(s^2-4s+8)^3}$

9. $\frac{16}{(s^2+4)^2}$ 10. $\frac{2(s-1)}{(s^2-2s+5)^2}+\frac{2(s+1)}{(s^2+2s+5)^2}$ 11. $\frac{2(s+1)}{s(s^2+2s+2)^2}$

12. $\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$ 13. $\log\left(\frac{s+2}{s+1}\right)$ 14. $\frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)$ 15. $\cot^{-1}\left(\frac{s+3}{2}\right)$

16. $s\log\left(\frac{s}{\sqrt{s^2+1}}\right)+\cot^{-1}s$ 17. $\frac{1}{s+1}\cot^{-1}(s+3)$

18. $\frac{2s^3+20s^2+64s+100}{s^2(s^2+8s+25)^2}+\cot^{-1}\left(\frac{s}{5}\right)$

Page No. 20

1. $\frac{2}{25}$ 2. $\frac{\pi}{4}$ 3. $\log 2$

Page No. 22

Initial value = 0, Final value = 0.

Page No. 28

1. $\frac{1}{s}\tanh\left(\frac{sb}{2}\right)$

Page No. 32

1. $\frac{t^2 e^{4t}}{2}$ 2. $\frac{e^{bt}}{a}(a\cos at + b\sin at)$ 3. $\frac{e^{-3t}}{2}(2\cos 2t - 3\sin 2t)$

4. $e^{-t}(\cos t - 3\sin t)$ 5. $e^{-t}\cosh t$ 6. $e^{-t}\cos 3t$

7. $e^{-3t}(\cosh 5t - \sinh 5t)$ 8. $e^{-\frac{t}{2}}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$

9. $\frac{e^{-at}}{b}[bc\cos bt + (d-ac)\sin bt]$

Page No. 33

1. $\frac{t\sinh t}{2}$ 2. $\frac{t\sin 2t}{4}$ 3. $\frac{t e^{-3t}\sin 2t}{4}$ 4. $t e^{-t}\sin t$

Page No. 35

1. $\frac{2\sinh t}{t}$ 2. $\frac{1-e^{-t}}{t}$ 3. $\frac{2\cos t - 1}{t}$ 4. $\frac{\sin t}{t}$

Page No. 37

1. $\frac{t^3 e^{-3t}(4-3t)}{24}$ 2. $\frac{t e^{t}}{6}(t^2+6t+6)$

Page No. 39

1. $\frac{1-\cos at}{a^2}$ 2. $\frac{e^{at}-1}{a}$ 3. $-\frac{1}{8}\left[e^{-2t}(2t^2+2t+1)-1\right]$

Page No. 45

1. $-1+t+e^{-t}$ 2. $e^{-t}(1-\cos t)$

3. $\frac{1}{2}(e^{-t} - \cos t + \sin t)$ 4. $-\frac{1}{3} + \frac{8e^{3t}}{15} + \frac{4e^{-2t}}{5}$ 5. $\frac{1}{3}(\cos t - \cos 2t)$

6. $-1 + e^{t} - \frac{\sin 2t}{2}$

7. $\frac{1}{5}\left(1 - e^{t}\cos 2t + \frac{e^{t}\sin 2t}{2}\right)$ 8. $\frac{1}{5}(e^{2t} - \cos t - 2\sin t)$

Page No. 47

1. $\frac{1-\cos 3(t-2)}{9},\ t>2$ 2. $\frac{-\sin(t-3)}{t-3},\ t>3$

Page No. 51

1. $(a)\ \frac{t}{25}-\frac{\sin 5t}{125}$ $\quad (b)\ \frac{t^2}{10}-\frac{t}{25}+\frac{1}{125}-\frac{e^{-5t}}{125}$ $\quad (c)\ \frac{\cosh at-1}{a^2}$

$(d)\ \frac{e^{-t}}{4}(\sin 2t-2t\cos 2t)$ $\quad$ 2. $\frac{1}{2a^3}(\sin at-at\cos at)$

Page No. 58

1. $y=-2e^{-t}+te^{-t}+2e^{-2t}$ $\quad$ 2. $y=-1-\frac{e^{-5t}}{6}+\frac{7e^{t}}{6}$

3. $y=e^{-t}(1-\cos 2t)$ 4. $y=1-e^{-x}\cos x$ $\quad$ 5. $y=e^{-t}-e^{-2t}-te^{-2t}$

MA 242303 TRANSFORMS & RANDOM PROCESSES
UNIT II – FOURIER TRANSFORM

Fourier integral theorem.

If f(x) is a given function defined in ($-l, l$) and satisfies Dirichlet's conditions then

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(x-t)}\, dt\, d\lambda \quad (or) \quad f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos[\lambda(x-t)]\, dt\, d\lambda$$

Definition:
Fourier transform pair.

Fourier transform of f(x) is defined as $F(s)=F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

Its Inverse Fourier transform is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds = F^{-1}[F(s)]$

Fourier cosine transform pair.

Fourier cosine transform of f(x) is $F_c(s)=F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\, dx$

Its Inverse Fourier cosine transform is $f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c[f(x)]\cos sx\, ds$

Fourier sine transform pair.

Fourier sine transform of f(x) is $F_s(s)=F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\, dx$

Its Inverse Fourier sine transform is $f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s[f(x)]\sin sx\, ds$

Parseval's identity for Fourier transform.

If F(s) is the Fourier transform of f(x) then $\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

Parseval's identity for Fourier sine and cosine transform.

i) If F$_s$(s) and F$_c$(s) are the Fourier sine and Fourier cosine transform of f(x) respectively then

$$\int_0^{\infty}[F_s(s)]^2 ds=\int_0^{\infty}[f(x)]^2\, dx \quad and \quad \int_0^{\infty}[F_c(s)]^2 ds=\int_0^{\infty}[f(x)]^2\, dx$$

ii) If F$_s$(s) and F$_c$(s) are the Fourier sine and Fourier cosine transform of f(x) and g(x) respectively then

$$\int_0^{\infty}F_s(s)\, G_s(s)\, ds=\int_0^{\infty} f(x)\, g(x)\, dx \quad and \quad \int_0^{\infty}F_c(s)\, G_c(s)\, ds=\int_0^{\infty} f(x)\, g(x)\, dx$$

Note:

i) $F_s[x\, f(x)]=-\frac{d}{ds}F_c[f(x)]$ ii) $F_c[x\, f(x)]=\frac{d}{ds}F_s[f(x)]$

iii) $F[x^n\, f(x)]=(-i)^n\frac{d^n}{ds^n}F[f(x)]$

Problems

1. Find the Fourier transform of $f(x)=\begin{cases} a^2-x^2\,, & |x|<a \\ 0\,, & |x|\geq a \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin s - s\cos s}{s^3}\,ds=\frac{\pi}{4}$ $\quad (ii)\ \int_0^\infty \frac{\sin s - s\cos s}{s^3}\cos\frac{s}{2}\,ds=\frac{3\pi}{16}$

$(iii)\ \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds=\frac{\pi}{15}$

Sol.

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0.e^{isx}\,dx+\int_{-a}^{a}(a^2-x^2)\,e^{isx}\,dx+\int_{a}^{\infty}0.e^{isx}\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\cos sx\,dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)\sin sx\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}(a^2-x^2)\cos sx\,dx+0$$

$$=\sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{2a\cos as}{s^2}+\frac{2\sin as}{s^3}\right\}-\{0-0+0\}\right]$$

$(i.e.)\ F[f(x)]=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$

When a = 1, we have $F[f(x)]=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

Using ***inverse Fourier transform***, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F[f(x)]\,e^{-isx}\,ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx-i\sin sx)\,ds$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds-i\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\,ds$$

$$=\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds-0$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds=\frac{\pi}{4}f(x)\ -------\ (1)$$

Put $x=0$ *in equation* (1) *we get*

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds=\frac{\pi}{4}f(0)$$

$$=\frac{\pi}{4}(1)=\frac{\pi}{4}\quad \textit{This proves } (i)$$

$f(x) = a^2 - x^2$
$f(x) = 1 - x^2$
$f(0) = 1 - 0 = 1$

$f(x) = a^2 - x^2$
$f(x) = 1 - x^2$
$f(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

Put $x = \frac{1}{2}$ *in equation* (1) *we get*

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos\frac{s}{2}\,ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{4}\right) = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16} \quad \textit{This proves } (ii)$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1}(1-x^2)^2\,dx + \int_1^\infty 0.dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1}(1-x^2)^2\,dx$$

$$\frac{16}{\pi}\int_0^\infty\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_0^1 (1-x^2)^2\,dx$$

$$= 2\int_0^1 (1 + x^4 - 2x^2)\,dx$$

$$= 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[\left\{1 + \frac{1}{5} - \frac{2}{3}\right\} - \{0 + 0 - 0\}\right]$$

$$= 2\left[\frac{8}{15}\right]$$

$$\frac{16}{\pi}\int_0^\infty\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\int_0^\infty\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15} \quad \textit{This proves } (iii)$$

2. Find the Fourier transform of $f(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$

Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

Sol. $$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0.e^{isx}\,dx + \int_{-1}^{1}(1-|x|)\,e^{isx}\,dx + \int_1^\infty 0.e^{isx}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos sx\,dx + i\,\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\sin sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-|x|)\cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{0 - \frac{\cos s}{s^2}\right\} - \left\{0 - \frac{1}{s^2}\right\}\right]$$

$$(i.e.)\quad F[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2}\right]$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]\right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1-|x|)^2\, dx + \int_{1}^{\infty} 0.dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1} (1-|x|)^2\, dx$$

$$\frac{4}{\pi}\int_{0}^{\infty} \left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_{0}^{1} (1-x)^2\, dx$$

$$\frac{4}{\pi}\int_{0}^{\infty} \left(\frac{1-\cos 2t}{4t^2}\right)^2 2dt = 2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

Put s = 2t
ds = 2dt

$$\frac{8}{16\pi}\int_{0}^{\infty} \left(\frac{1-\cos 2t}{t^2}\right)^2 dt = 2\left[\{0\} - \left\{-\frac{1}{3}\right\}\right]$$

$$\frac{1}{2\pi}\int_{0}^{\infty} \left(\frac{2\sin^2 t}{t^2}\right)^2 dt = \frac{2}{3}$$

$$\frac{4}{2\pi}\int_{0}^{\infty} \left(\frac{\sin^2 t}{t^2}\right)^2 dt = \frac{2}{3}$$

$$(i.e.)\quad \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$$

3. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Hence deduce that $(i)\ \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$ $\qquad (ii)\ \int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

Sol. $$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0.e^{isx}\, dx + \int_{-1}^{1} (1)\, e^{isx}\, dx + \int_{1}^{\infty} 0.e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \cos sx \, dx + i \, \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \sin sx \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} \cos sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} - 0 \right]$$

$$(i.e.) \quad F[f(x)] = \sqrt{\frac{2}{\pi}} \, \frac{\sin s}{s}$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)] \, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin s}{s} \right) (\cos sx - i \sin sx) \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s} \right) \cos sx \, ds - i \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s} \right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s} \right) \cos sx \, ds - 0$$

$$\int_{0}^{\infty} \left(\frac{\sin s}{s} \right) \cos sx \, ds = \frac{\pi}{2} f(x)$$

Put $x = 0$ *we get*

$$\int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} f(0)$$

$$= \frac{\pi}{2} (1)$$

$f(x) = 1$
$f(0) = 1$

$$(i.e.) \quad \int_{0}^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2 \, ds = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \, \frac{\sin s}{s} \right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1)^2 \, dx + \int_{1}^{\infty} 0.dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s}{s} \right)^2 ds = \int_{-1}^{1} dx$$

$$\frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s} \right)^2 ds = [x]_{-1}^{1}$$

$$= 1 - (-1)$$

$$\frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin s}{s} \right)^2 ds = 2 \quad \Rightarrow \quad \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

4. Find the sine transform of $\frac{1}{x}$

Sol. $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_s\left[\frac{1}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{t}\sin t\,\frac{dt}{s}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}\,dt$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2}$$

$$= \sqrt{\frac{\pi}{2}}$$

Put $sx = t$
$s\,dx = dt$

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$$

5. Find f(x) if its sine transform is e^{-as}

Sol. The inverse Fourier sine transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-as}\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-as}}{a^2+x^2}(-a\sin sx - x\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{a^2+x^2}(0-x)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{x}{x^2+a^2}$$

6. Find f(x) if its cosine transform is $f_c(p) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a-\frac{s}{2}\right), & s < 2a \\ 0, & s \geq 2a \end{cases}$

Sol. The inverse Fourier cosine transform is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\cos sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^{2a}\frac{1}{\sqrt{2\pi}}\left(a-\frac{s}{2}\right)\cos sxds + \int_{2a}^\infty 0ds\right]$$

$$= \frac{1}{\pi}\left[\left(a-\frac{s}{2}\right)\left(\frac{\sin sx}{x}\right) - \left(-\frac{1}{2}\right)\left(\frac{-\cos sx}{x^2}\right)\right]_0^{2a}$$

$$= \frac{1}{\pi}\left[\left\{0 - \frac{\cos 2ax}{2x^2}\right\} - \left\{0 - \frac{1}{2x^2}\right\}\right]$$

$$= \frac{1}{\pi x^2}\,\frac{1-\cos 2ax}{2}$$

$$= \frac{\sin^2 ax}{\pi x^2}$$

7. Find the Fourier sine and cosine transform of e^{-ax}

Sol. $$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{a^2+s^2}(0-s)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

8. Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}$

Sol. $$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin sx\,dx$$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

Diff. w.r.t.'s' on both sides we get

$$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\sin sx\right)dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx.x\;dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\;dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$\frac{d}{ds}F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\;\frac{a}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int\frac{a}{s^2+a^2}ds$$

$$=a\sqrt{\frac{2}{\pi}}\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$\boxed{\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$$

$$=\sqrt{\frac{2}{\pi}}\;\tan^{-1}\left(\frac{s}{a}\right)$$

9. Find the Fourier cosine transform of $\frac{e^{-ax}}{x}$

Sol.

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$F_c\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx$$

Diff. w.r.t.'s' on both sides we get

$$\frac{d}{ds}F_c\left[\frac{e^{-ax}}{x}\right]=\frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}(-\sin sx.x)\;dx$$

$$=-\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\;\sin sx\;dx$$

$$=-\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx-s\cos sx)\right]_0^\infty$$

$$=-\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(0-s)\right\}\right]$$

$$\frac{d}{ds}F_c\left[\frac{e^{-ax}}{x}\right]=-\sqrt{\frac{2}{\pi}}\;\frac{s}{s^2+a^2}$$

Integrating w.r.t. 's' we get

$$F_c\left[\frac{e^{-ax}}{x}\right]=-\sqrt{\frac{2}{\pi}}\int\frac{s}{s^2+a^2}ds$$

$$\boxed{\int\frac{x\,dx}{x^2+a^2}=\frac{1}{2}\log(x^2+a^2)}$$

$$=-\sqrt{\frac{2}{\pi}}\;\frac{1}{2}\;\log(s^2+a^2)$$

$$=-\frac{1}{\sqrt{2\pi}}\;\log(s^2+a^2)$$

10. Find the Fourier sine and cosine transform of $x e^{-ax}$

Sol.

$$F_s[x e^{-ax}] = -\frac{d}{ds} F_c[e^{-ax}]$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2}(-a+0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

$$F_s[x e^{-ax}] = -\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \right]$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{-a}{(s^2+a^2)^2}(2s) \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2+a^2)^2}$$

$$F_c[x e^{-ax}] = \frac{d}{ds} F_s[e^{-ax}]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{a^2+s^2}(0-s) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}$$

$$F_c[x e^{-ax}] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2-s^2}{(s^2+a^2)^2}$$

11. Solve the integral equation $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$

Sol. Given $\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx=\sqrt{\frac{2}{\pi}}\ e^{-\lambda}$$

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\ e^{-\lambda}$$

$$f(x)=F_c^{-1}\left[\sqrt{\frac{2}{\pi}}\ e^{-\lambda}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}e^{-\lambda}\cos\lambda x\,d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\,d\lambda$$

$$=\frac{2}{\pi}\left[\frac{e^{-\lambda}}{1+x^2}(-\cos\lambda x+\lambda\sin\lambda x)\right]_0^\infty$$

$$=\frac{2}{\pi}\left[\{0\}-\left\{\frac{1}{1+x^2}(-1+0)\right\}\right]$$

$$(i.e.)\,f(x)=\frac{2}{\pi}\,\frac{1}{1+x^2}$$

12. Find the Fourier transform of $e^{-\frac{x^2}{2}}$

Sol. $F[f(x)]=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx$

$$F\left[e^{-\frac{x^2}{2}}\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}\,e^{isx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left[x^2-2isx\right]}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left[(x-is)^2-\ i^2s^2\right]}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left[(x-is)^2\right]}\,e^{-\frac{s^2}{2}}\,dx$$

$$=\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x-is)^2}\,dx$$

$$=\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}\,dx$$

$$\boxed{\begin{aligned}Put\ \ \frac{x-is}{\sqrt{2}}&=t\\ \frac{dx}{\sqrt{2}}&=dt\end{aligned}}$$

$$=\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-t^2}\ \sqrt{2}\ dt$$

$$\boxed{\int_{-\infty}^{\infty}e^{-t^2}dt=\sqrt{\pi}}$$

$$=\frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-t^2}\ dt$$

$$=\frac{e^{-\frac{s^2}{2}}}{\sqrt{\pi}}\ \sqrt{\pi}\ =e^{-\frac{s^2}{2}}$$

(i.e.) $F[e^{-\frac{x^2}{2}}]=e^{-\frac{s^2}{2}}$

Note: If the transform of f(x) is equal to f(s), then the function f(x) is called self-reciprocal. In the above problem, $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform.

13. Find the Fourier cosine transform of $e^{-a^2x^2}$ and hence find $F_s[x e^{-a^2x^2}]$

Sol.

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx$$

$$F_c\left[e^{-a^2x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2} \cos sx\ dx$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{1}{2} \int_{-\infty}^\infty e^{-a^2x^2} \cos sx\ dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^\infty e^{-a^2x^2}\ e^{isx}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^\infty e^{-\left[a^2x^2 - isx\right]}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^\infty e^{-\left[\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right]}\ dx$$

$$= \frac{1}{\sqrt{2\pi}} R.P. \int_{-\infty}^\infty e^{-\left(ax - \frac{is}{2a}\right)^2}\ e^{-\frac{s^2}{4a^2}}\ dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} R.P. \int_{-\infty}^\infty e^{-\left(ax - \frac{is}{2a}\right)^2}\ dx$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{\sqrt{2\pi}} R.P. \int_{-\infty}^\infty e^{-t^2}\ \frac{dt}{a}$$

$$\boxed{\begin{aligned} Put\ \ ax - \frac{is}{2a} &= t \\ a\ dx &= dt \end{aligned}}$$

$$= \frac{e^{-\frac{s^2}{4a^2}}}{a\sqrt{2\pi}} R.P.\ \sqrt{\pi}$$

$$(i.e.)\ \ F_c[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

$$F_s[x e^{-a^2x^2}] = -\frac{d}{ds} F_c[e^{-a^2x^2}]$$

$$= -\frac{d}{ds}\left[\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}\right]$$

$$= -\frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}} \left(\frac{-2s}{4a^2}\right) = \frac{s}{2\sqrt{2}\, a^3} e^{-\frac{s^2}{4a^2}}$$

14. Find the Fourier cosine transform of e^{-4x}. Hence deduce that $\int_0^\infty \frac{\cos 2x}{x^2+16}\,dx = \frac{\pi}{8}e^{-8}$ and $\int_0^\infty \frac{x\sin 2x}{x^2+16}\,dx = \frac{\pi}{2}e^{-8}$

Sol. $F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$

$$F_c[e^{-4x}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-4x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-4x}}{16+s^2}(-4\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\{0\} - \left\{\frac{1}{16+s^2}(-4+0)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{4}{s^2+16}$$

Using ***inverse Fourier cosine transform***, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c[f(x)]\,\cos sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{4}{s^2+16}\right)\cos sx\,ds$$

$$f(x) = \frac{8}{\pi}\int_0^\infty \frac{\cos sx}{s^2+16}\,ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+16}\,ds = \frac{\pi}{8}f(x)$$

$$\int_0^\infty \frac{\cos sx}{s^2+16}\,ds = \frac{\pi}{8}e^{-4x} \text{ --------- } (1)$$

Put $x = 2$, *we get*

$$\int_0^\infty \frac{\cos 2s}{s^2+16}\,ds = \frac{\pi}{8}e^{-8}$$

$$\int_0^\infty \frac{\cos 2x}{x^2+16}\,dx = \frac{\pi}{8}e^{-8}$$

Differentiate (1) w.r.t. x, we get

$$\frac{d}{dx}\int_0^\infty \frac{\cos sx}{s^2+16}\,ds = \frac{\pi}{8}\frac{d}{dx}(e^{-4x})$$

$$\int_0^\infty \frac{\partial}{\partial x}\left(\frac{\cos sx}{s^2+16}\right)ds = \frac{\pi}{8}\frac{d}{dx}(e^{-4x})$$

$$\int_0^\infty \left(\frac{-\sin sx.s}{s^2+16}\right)ds = \frac{\pi}{8}(e^{-4x})(-4)$$

$$\int_0^\infty \frac{s\,\sin sx}{s^2+16}\,ds = \frac{\pi}{2}e^{-4x}$$

$$Put \quad x = 2, \quad we \quad get$$

$$\int_0^\infty \frac{s \sin 2s}{s^2 + 16} \; ds = \frac{\pi}{2} e^{-8}$$

$$\int_0^\infty \frac{x \sin 2x}{x^2 + 16} \; dx = \frac{\pi}{2} e^{-8}$$

15. Find the Fourier sine and cosine transform of e^{-x} and hence find the Fourier sine transform of $\frac{x}{1+x^2}$ and Fourier cosine transform of $\frac{1}{1+x^2}$

Sol.
$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{1+s^2} (-1+0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \; \frac{1}{s^2+1}$$

$$F_s[e^{-x}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[\{0\} - \left\{ \frac{1}{1+s^2} (0-s) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \; \frac{s}{s^2+1}$$

Now, $F_c\left[\frac{1}{1+x^2}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{1+x^2} \cos sx \, dx \quad ----- (1)$

Using ***inverse Fourier cosine transform***, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \; \cos sx \, ds$$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{1}{s^2+1} \right) \cos sx \; ds$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\cos sx}{s^2+1} \; ds$$

$$\int_0^\infty \frac{\cos sx}{s^2+1} \; ds = \frac{\pi}{2} e^{-x}$$

$$\int_0^\infty \frac{\cos sx}{x^2+1} \; dx = \frac{\pi}{2} e^{-s}$$

Put $x = \text{s}$
and $\text{s} = x$

Equation (1) becomes

$$F_c\left[\frac{1}{1+x^2}\right]=\sqrt{\frac{2}{\pi}}\,\frac{\pi}{2}\,e^{-s}$$

$$=\sqrt{\frac{\pi}{2}}\,e^{-s}$$

$$F_s\left[\frac{x}{1+x^2}\right]=-\frac{d}{ds}F_c\left[\frac{1}{1+x^2}\right]$$

$$=-\frac{d}{ds}\left[\sqrt{\frac{\pi}{2}}\,e^{-s}\right]$$

$$=-\sqrt{\frac{\pi}{2}}\,e^{-s}(-1)$$

$$=\sqrt{\frac{\pi}{2}}\,e^{-s}$$

16. Find the Fourier transform of $f(x)=e^{-a|x|}$, $a>0$**. Hence deduce that** $(i)\ \int_0^\infty \frac{\cos xt}{a^2+t^2}dt=\frac{\pi}{2a}e^{-a|x|}$ $(ii)\ \int_0^\infty \frac{dx}{x^2+1}=\frac{\pi}{2}$ $(iii)\ \int_0^\infty \frac{dx}{(x^2+1)^2}=\frac{\pi}{4}$ **and also prove that** $(iv)\ F\left[x\,e^{-a|x|}\right]=i\sqrt{\frac{2}{\pi}}\,\frac{2as}{(s^2+a^2)^2}$

Sol.

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}\cos sx\,dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a|x|}\sin sx\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-ax}\cos sx\,dx+0$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx+s\sin sx)\right]_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[\{0\}-\left\{\frac{1}{a^2+s^2}(-a+0)\right\}\right]$$

$$(i.e.)\ F[f(x)]=\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$$

Using ***inverse Fourier transform***, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F[f(x)]\,e^{-isx}\,ds$$

$$e^{-a|x|}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)(\cos sx-i\sin sx)\,ds$$

$$=\frac{a}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{s^2+a^2}\right)\cos sx\,ds-i\frac{a}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{s^2+a^2}\right)\sin sx\,ds$$

$$=\frac{2a}{\pi}\int_0^\infty\frac{\cos sx}{s^2+a^2}\,ds-0$$

$$\int_0^\infty \frac{\cos sx}{s^2+a^2}ds = \frac{\pi}{2a}e^{-a|x|}$$

$$(i.e.)\ \int_0^\infty \frac{\cos xt}{t^2+a^2}dt = \frac{\pi}{2a}e^{-a|x|} \quad This\ \ proves\ \ (i)$$

$Put\ \ x=0\ \ and\ \ a=1,\ we\ \ get$

$$\int_0^\infty \frac{1}{t^2+a^2}\,dt = \frac{\pi}{2}$$

$$(i.e.)\ \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2} \quad This\ \ proves\ \ (ii)$$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}\right)^2 ds = \int_{-\infty}^{-\infty}\left[e^{-a|x|}\right]^2\ dx$$

$$\frac{2a^2}{\pi}\int_{-\infty}^{\infty}\frac{1}{(s^2+a^2)^2}ds = \int_{-\infty}^{-\infty}\left[e^{-a|x|}\right]^2\ dx$$

$$\frac{4a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2} = 2\int_0^\infty e^{-2ax}\,dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \frac{1}{-2a}[0-1]$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$\int_0^\infty \frac{ds}{(s^2+a^2)^2} = \frac{\pi}{4a^3}$$

$put\ \ a=1,\ \ we\ \ get$

$$\int_0^\infty \frac{ds}{(s^2+1)^2} = \frac{\pi}{4}$$

$$(i.e.)\ \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} \qquad This\ \ proves\ \ (iii)$$

By the property, $F[x\,f(x)] = (-i)\dfrac{d}{ds}F[f(x)]$

$$F[x e^{-a|x|}] = (-i)\frac{d}{ds}F[e^{-a|x|}]$$

$$= (-i)\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\ \frac{a}{s^2+a^2}\right]$$

$$= (-i)\sqrt{\frac{2}{\pi}}\left[\frac{-a}{(s^2+a^2)^2}\ (2s)\right]$$

$$= i\ \sqrt{\frac{2}{\pi}}\ \frac{2as}{(s^2+a^2)^2} \qquad This\ \ proves\ \ (iv)$$

17. Find the Fourier sine and cosine transform of $x^{n-1}, 0< n < 1, x > 0$ and hence prove that $\frac{1}{\sqrt{x}}$ is self reciprocal under both Fourier sine and cosine transforms. Also find $F\left[\frac{1}{\sqrt{|x|}}\right]$.

Sol. Consider $F_c[f(x)] - i\,F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx - i\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$

$$F_c[f(x)] - i\,F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)(\cos sx - i\sin sx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,e^{-isx}\,dx$$

$$F_c[x^{n-1}] - i\,F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\,e^{-isx}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{(is)^n}$$

$$= (-i)^n\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}$$

$$= \left(\cos\frac{n\pi}{2} - i\,\sin\frac{n\pi}{2}\right)\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}$$

$$\boxed{\int_0^\infty x^{n-1}\,e^{-ax}\,dx = \frac{\Gamma(n)}{a^n}}$$

$$\boxed{\begin{aligned} -i &= \cos\frac{\pi}{2} - i\,\sin\frac{\pi}{2} \\ (-i)^n &= \left(\cos\frac{\pi}{2} - i\,\sin\frac{\pi}{2}\right)^n \\ &= \cos\frac{n\pi}{2} - i\,\sin\frac{n\pi}{2} \end{aligned}}$$

Equating R.P and I.P, we get

$$F_c[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad -------- \ (1)$$

$$F_s[x^{n-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2} \quad -------- \ (2)$$

Put $n = \frac{1}{2}$ *in equation* (1), *we have*

$$F_c[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(1/2)}{s^{1/2}}\cos\frac{\pi}{4}$$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{s}}$$

Put $n = \frac{1}{2}$ *in equation* (1), *we have*

$$F_s[x^{\frac{1}{2}-1}] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(1/2)}{s^{1/2}}\sin\frac{\pi}{4}$$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{\pi}}{\sqrt{s}}\,\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{s}}$$

Hence $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier sine and cosine transforms.

Now, $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$F\left[\frac{1}{\sqrt{|x|}}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} \cos sx\, dx + i\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{|x|}} \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{x}} \cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{x}} \cos sx\, dx$$

$$= F_c\left[\frac{1}{\sqrt{x}}\right]$$

$$= \frac{1}{\sqrt{s}}$$

18. Verify Parseval's theorem of Fourier transform for the function $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$

Sol. $F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} 0.e^{isx}\, dx + \int_{0}^{\infty} e^{-x}.e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-is)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-is)x}}{-(1-is)} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{1}{-(1-is)} \right]$$

(*i.e.*) $F(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{1-is}$

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} F(s)\overline{F(s)}\, ds = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{1-is} \frac{1}{\sqrt{2\pi}} \frac{1}{1+is} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+s^2} ds$$

$$= \frac{2}{2\pi} \int_{0}^{\infty} \frac{ds}{1+s^2}$$

$$= \frac{1}{\pi} \left[\frac{1}{1} \tan^{-1}\left(\frac{s}{1}\right) \right]_0^{\infty}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{1}{2}$$

$$\int_{-\infty}^{\infty} | f(x) |^2 \, dx = \int_{-\infty}^{0} 0. \, dx + \int_{0}^{\infty} (e^{-x})^2 \, dx$$

$$= \int_{0}^{\infty} e^{-2x} \, dx$$

$$= \left[\frac{e^{-2x}}{-2} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-2} \right]$$

$$= \frac{1}{2}$$

$$\therefore \int_{-\infty}^{\infty} | F(s) |^2 \, ds = \int_{-\infty}^{\infty} | f(x) |^2 \, dx$$

Hence Parseval's theorem is verified.

19. Using Parseval's identity, calculate $i) \int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$ $ii) \int_0^{\infty} \frac{x^2 dx}{(x^2+4)^2}$

Sol. (i) Let $f(x) = e^{-ax}$ then $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$

Using Parseval's identity for Fourier cosine transform, we have

$$\int_0^{\infty} [F_c(s)]^2 \, ds = \int_0^{\infty} [f(x)]^2 \, dx$$

$$\int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2} \right)^2 ds = \int_0^{\infty} (e^{-ax})^2 \, dx$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \int_0^{\infty} e^{-2ax} \, dx$$

$$= \left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-2a} \right]$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2} = \frac{1}{2a}$$

$$(i.e.) \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$$

(ii) Let $f(x) = e^{-2x}$ then $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+4}$

Using Parseval's identity for Fourier sine transform, we have

$$\int_0^{\infty} [F_s(s)]^2 \, ds = \int_0^{\infty} [f(x)]^2 \, dx$$

$$\int_0^{\infty} \left(\sqrt{\frac{2}{\pi}} \frac{s}{s^2+4} \right)^2 ds = \int_0^{\infty} (e^{-2x})^2 \, dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2 ds}{(s^2+4)^2} = \int_0^{\infty} e^{-4x} \, dx$$

$$= \left[\frac{e^{-4x}}{-4} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-4} \right]$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2 ds}{(s^2+4)^2} = \frac{1}{4}$$

$$(i.e.) \int_0^{\infty} \frac{x^2 dx}{(x^2+4)^2} = \frac{\pi}{8}$$

20. Use transform methods to evaluate $i) \displaystyle\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$ $\quad ii) \displaystyle\int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+25)}$

Sol. (i) Let $f(x) = e^{-x}$ and $g(x) = e^{-2x}$

Then $F_c(s) = F_c[f(x)] = \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1}$ and $G_c(s) = G_c[g(x)] = \sqrt{\frac{2}{\pi}} \frac{2}{s^2+4}$

We have $\displaystyle\int_0^{\infty} F_c(s) G_c(s)\, ds = \int_0^{\infty} f(x)\, g(x)\, dx$

$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1} \sqrt{\frac{2}{\pi}} \frac{2}{s^2+4} ds = \int_0^{\infty} e^{-x} e^{-2x}\, dx$$

$$\frac{4}{\pi} \int_0^{\infty} \frac{ds}{(s^2+1)(s^2+4)} = \int_0^{\infty} e^{-3x}\, dx$$

$$= \left[\frac{e^{-3x}}{-3} \right]_0^{\infty}$$

$$= \left[0 - \frac{1}{-3} \right]$$

$$\frac{4}{\pi} \int_0^{\infty} \frac{ds}{(s^2+1)(s^2+4)} = \frac{1}{3}$$

$$(i.e.) \int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12}$$

(ii) Let $f(x) = e^{-3x}$ and $g(x) = e^{-5x}$

Then $F_s(s) = F_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+9}$ and $G_s(s) = G_s[g(x)] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2+25}$

We have $\displaystyle\int_0^{\infty} F_s(s) G_s(s)\, ds = \int_0^{\infty} f(x)\, g(x)\, dx$

$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+9} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+25} ds = \int_0^{\infty} e^{-3x} e^{-5x}\, dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2 ds}{(s^2+9)(s^2+25)} = \int_0^{\infty} e^{-8x}\, dx$$

$$= \left[\frac{e^{-8x}}{-8} \right]_0^{\infty} = \left[0 - \frac{1}{-8} \right]$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2 ds}{(s^2+9)(s^2+25)} = \frac{1}{8} \quad \Rightarrow \quad \int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+25)} = \frac{\pi}{16}$$

21. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ **using transforms.**

Sol. Let $f(x)=e^{-ax}$ and $g(x)=e^{-bx}$

Then $F_c(s)=F_c[f(x)]=\sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}$ and $G_c(s)=G_c[g(x)]=\sqrt{\frac{2}{\pi}}\,\frac{b}{s^2+b^2}$

We have $\int_0^\infty F_c(s)G_c(s)\,ds=\int_0^\infty f(x)\,g(x)\,dx$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{a}{s^2+a^2}\,\sqrt{\frac{2}{\pi}}\,\frac{b}{s^2+b^2}ds=\int_0^\infty e^{-ax}\,e^{-bx}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}=\int_0^\infty e^{-(a+b)x}\,dx$$

$$=\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty=\left[0-\frac{1}{-(a+b)}\right]$$

$$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}=\frac{1}{a+b}$$

$$(i.e.)\ \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{\pi}{2ab(a+b)}$$

22. Find the Fourier sine transform of $f(x)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

Sol. $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\sin sx\,dx$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\,dx+\int_1^2(2-x)\sin sx\,dx+\int_2^\infty 0.\sin sx\,dx\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[x\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1+\sqrt{\frac{2}{\pi}}\left[(2-x)\left(\frac{-\cos sx}{s}\right)-(-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2$$

$$=\sqrt{\frac{2}{\pi}}\left[\left\{\frac{-\cos s}{s}+\frac{\sin s}{s^2}\right\}-\{0+0\}\right]+\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{\sin 2s}{s^2}\right\}-\left\{\frac{-\cos s}{s}-\frac{\sin s}{s^2}\right\}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s}{s^2}-\frac{\sin 2s}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s-2\sin s\cos s}{s^2}\right]$$

$$=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s\,(1-\cos s)}{s^2}\right]$$

Properties of Fourier Transform

1. Prove that F[af(x) + bg(x)] = aF(s) + bG(s) [Linearity property on Fourier transform]

Proof. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[a\,f(x) + b\,g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [a\,f(x) + b\,g(x)]\, e^{isx}\, dx$$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \; + \; b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx$$

$$= a\,F(s) + b\,G(s)$$

2. Prove (i)F$_c$[af(x) + bg(x)] = aF$_c$(s) + bG$_c$(s)[Linear property on Fourier cosine transform]

(ii)F$_s$[af(x) + bg(x)] = aF$_s$(s) + bG$_s$(s)[Linear property on Fourier sine transform]

Proof. (i) $F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos sx\, dx = F_c(s)$

$$F_c[a\,f(x) + b\,g(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [a\,f(x) + b\,g(x)]\cos sx\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\cos sx\, dx + b\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x)\cos sx\, dx$$

$$= a\,F_c(s) + b\,G_c(s)$$

(ii) $F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin sx\, dx = F_s(s)$

$$F_s[a\,f(x) + b\,g(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} [a\,f(x) + b\,g(x)]\sin sx\, dx$$

$$= a\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin sx\, dx + b\sqrt{\frac{2}{\pi}} \int_0^{\infty} g(x)\sin sx\, dx$$

$$= a\,F_s(s) + b\,G_s(s)$$

3. Prove that $F[f(x-a)] = e^{ias}F(s)$ [Time shifting property]

Proof. We have $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

Put $x - a = t$
$dx = dt$

$$= e^{ias}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{ias}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= e^{ias}\,F(s)$$

4. Prove that $F[e^{iax} f(x)] = F(s+a)$ **[Frequency shifting property]**

Proof. We have $F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\, dx$$

$$= F(s+a)$$

5. Prove that $(i)\;\; F[f(ax)] = \dfrac{1}{a} F\left(\dfrac{s}{a}\right),\; a > 0$ **[Change of scale property]**

$$(ii)\;\; F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$$

$$(iii)\;\; F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

Proof. (i) We have $F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = F(s)$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is\frac{t}{a}} \frac{dt}{a}$$

Put a x = t
a dx = dt

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\frac{s}{a}t}\, dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

(ii) We have $F_s[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_{0}^{\infty} f(x) \sin sx\, dx = F_s(s)$

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(ax) \sin sx\, dx$$

Put a x = t
a dx = dt

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin\left(\frac{st}{a}\right) \frac{dt}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin\left(\frac{s}{a}\right) t\, dt$$

$$= \frac{1}{a} F_s\left(\frac{s}{a}\right)$$

(iii) We have $F_c[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_{0}^{\infty} f(x) \cos sx\, dx = F_c(s)$

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(ax) \cos sx\, dx$$

Put a x = t
a dx = dt

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos\left(\frac{st}{a}\right) \frac{dt}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos\left(\frac{s}{a}\right) t\, dt \;= \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

6. If $\overline{f}(\lambda)$ is the Fourier transform of $f(x)$, find the Fourier transform of $f(x-a)$ and $f(ax)$.

Proof. $F[f(x-a)]=e^{ia\lambda}\,\overline{f}(\lambda)$ [see property (3) and (5) (*i*)]

and $F[f(ax)]=\frac{1}{a}\,\overline{f}\left(\frac{\lambda}{a}\right)$

7. Prove that [Modulation property]

(*i*) $F[f(x)\cos ax]=\frac{1}{2}[F(s+a)+F(s-a)]$ (*ii*) $F_s[f(x)\cos ax]=\frac{1}{2}[F_s(s+a)+F_s(s-a)]$

(*iii*) $F_s[f(x)\sin ax]=\frac{1}{2}[F_c(s-a)-F_c(s+a)]$ (*iv*) $F_c[f(x)\cos ax]=\frac{1}{2}[F_c(s+a)+F_c(s-a)]$

(*v*) $F_c[f(x)\sin ax]=\frac{1}{2}[F_s(a+s)+F_s(a-s)]$

Proof. (*i*) We have $F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$

$$F[f(x)\cos ax]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\; e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{isx}\,dx$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i(s+a)x}\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{i(s-a)x}\,dx\right]$$

$$=\frac{1}{2}[F(s+a)+F(s-a)]$$

(*ii*) We have $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\sin sx\,dx = F_s(s)$

2sinAcosB = sin(A + B) + sin(A – B)

$$F_s[f(x)\cos ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax\,\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}[\sin(s+a)x+\sin(s-a)x]\,dx$$

$$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\sin(s+a)x\,dx+\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\sin(s-a)x\,dx\right]$$

$$=\frac{1}{2}\,[F_s(s+a)+F_s(s-a)]$$

(*iii*) We have $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\sin sx\,dx = F_s(s)$

2sinAsinB = cos(A – B) – cos(A + B)

$$F_s[f(x)\sin ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax\,\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}[\cos(s-a)x-\cos(s+a)x]\,dx$$

$$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\cos(s-a)x\,dx-\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\cos(s+a)x\,dx\right]$$

$$=\frac{1}{2}\,[F_c(s-a)-F_c(s+a)]$$

(iv) We have $F_c[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \cos sx \, dx = F_c(s)$

$$F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos ax \cos sx \, dx$$

2cosAcosB = cos(A + B) + cos(A – B)

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, \frac{1}{2}[\cos(s+a)x + \cos(s-a)x] \, dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(s+a)x \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(s-a)x \, dx\right]$$

$$= \frac{1}{2}\,[F_c(s+a) + F_c(s-a)]$$

(v) We have $F_c[f(x)] = \sqrt{\dfrac{2}{\pi}} \displaystyle\int_0^\infty f(x) \cos sx \, dx = F_c(s)$

$$F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin ax \cos sx \, dx$$

2sinAcosB = sin(A + B) + sin(A – B)

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \, \frac{1}{2}[\sin(a+s)x + \sin(a-s)x] \, dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(a+s)x \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(a-s)x \, dx\right]$$

$$= \frac{1}{2}\,[F_s(a+s) + F_s(a-s)]$$

8. Prove that (i) $F[f(-x)] = F(-s)$ (ii) $F[\overline{f(x)}] = \overline{F(-s)}$ (iii) $F[\overline{f(-x)}] = \overline{F(s)}$

Proof. (i) We have $F[f(x)] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx = F(s)$

$$F[f(-x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)\, e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\, e^{-ist} \, (-dt)$$

Put $-x = t$
$-dx = dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i(-s)t} \, dt$$

$$= F(-s)$$

(ii) We have $F(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{isx} \, dx$

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx} \, dx$$

$$\overline{F(-s)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{isx} \, dx$$

$$= F[\overline{f(x)}]$$

(*iii*) We have $F(s)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$$\overline{F(s)}=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \overline{f(x)}\, e^{-isx}\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{\infty}^{-\infty} \overline{f(-t)}\, e^{ist}\, (-dt)$$

Put $-x=t$
$-dx=dt$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \overline{f(-t)}\, e^{ist}\, dt$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} \overline{f(-x)}\, e^{isx}\, dx$$

$$=F[\overline{f(-x)}]$$

Convolution of two functions for Fourier transform.

The convolution of two functions f(x) and g(x) is defined by

$$(f*g)(x)=f(x)*g(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\; g(x-t)\; dt$$

Convolution theorem

Statement. If F[f(x)] = F(s) and F[g(x)] = G(s) then $F[f(x)*g(x)]=F(s).G(s)$

Proof. $F[f(x)*g(x)]=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} [f(x)*g(x)]\, e^{isx}\, dx$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\, g(x-t)\; dt\right] e^{isx}\; dx$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(x-t)\; e^{isx}\; dx\right] dt$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(x-t)\; e^{isx}\, e^{ist}\, e^{-ist}\; dx\right] dt$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} g(x-t)\; e^{is(x-t)}\; d(x-t)\right] e^{ist}\; dt$$

$$=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\; G(s)\; e^{ist}\; dt$$

$$=G(s)\;\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(t)\; e^{ist}\; dt$$

$$=G(s)\,F(s)$$

(i.e.) $F[f(x)*g(x)]=F(s).G(s)$

Parseval's identity for Fourier transform.

Statement: If F(s) is the Fourier transform of f(x) then

$$\int\limits_{-\infty}^{\infty} |\,F(s)\,|^2\; ds=\int\limits_{-\infty}^{\infty} |\,f(x)\,|^2\; dx$$

Proof. By convolution theorem for Fourier transform, we have

$$F[f(x)*g(x)]=F(s).G(s)$$

$$\therefore\; F^{-1}[F(s)G(s)]=f(x)*g(x)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)G(s)\, e^{-isx}\, ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

$$\Rightarrow \int_{-\infty}^{\infty} F(s)G(s)\, e^{-isx}\, ds = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty} F(s)G(s)\, ds = \int_{-\infty}^{\infty} f(t)\, g(-t)\, dt \;------- (1)$$

$$Let\;\; g(-t) = \overline{f(t)} \;\;------- (2)$$

$$(i.e.)\;\; g(t) = \overline{f(-t)}$$

$$\begin{aligned} G(s) = F[g(x)] &= F[g(t)] \\ &= F[\overline{f(-t)}] \\ &= F[\overline{f(-x)}] \\ &= \overline{F(s)} \quad (by\;\; property) \end{aligned}$$

$$(i.e.)\;\; G(s) = \overline{F(s)} \;\;---------- (3)$$

Substituting (2) and (3) in equation (1) we have

$$\int_{-\infty}^{\infty} F(s)\overline{F(s)}\, ds = \int_{-\infty}^{\infty} f(t)\overline{f(t)}\, dt$$

(i.e.) $$\int_{-\infty}^{\infty} | F(s) |^2\, ds = \int_{-\infty}^{\infty} | f(x) |^2\, dx$$

University Questions:

1. State the Fourier transform of the derivatives of a function.

Sol. $F[f'(x)] = (-is)F(s)$

$F[f''(x)] = (-is)^2 F(s)$

$F[f'''(x)] = (-is)^3 F(s)$

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Ingeneral, $F[f^{(n)}(x)] = (-is)^n F(s)$

2. Give an example for self-reciprocal under Fourier transform.

Sol. $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier transform.

3. Give an example for self-reciprocal under Fourier cosine transform.

Sol. $e^{-\frac{x^2}{2}}$ is self-reciprocal under Fourier cosine transform.

4. Give an example for self-reciprocal under both Fourier sine and cosine transform.

Sol. $\frac{1}{\sqrt{x}}$ is self-reciprocal under both Fourier sine and cosine transform.

5. Find the Fourier transform of $f(x)=\begin{cases} a-|x|\,, & |x|<a \\ 0\ , & |x|\geq a \end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin t}{t}dt=\frac{\pi}{2} \qquad (ii)\ \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$

Sol.

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0.e^{isx}\, dx+\int_{-a}^{a}(a-|x|)\, e^{isx}\, dx+\int_{a}^{\infty} 0.e^{isx}\, dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)(\cos sx+i\sin sx)\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)\cos sx\, dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a-|x|)\sin sx\, dx$$

$$=\frac{2}{\sqrt{2\pi}}\int_{0}^{a}(a-|x|)\cos sx\, dx+0$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{a}(a-x)\cos sx\, dx$$

$$=\sqrt{\frac{2}{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[\left\{0-\frac{\cos sa}{s^2}\right\}-\left\{0-\frac{1}{s^2}\right\}\right]$$

$$(i.e.)\ F[f(x)]=\sqrt{\frac{2}{\pi}}\,\frac{1-\cos as}{s^2}$$

Using ***inverse Fourier transform***, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F[f(x)]\, e^{-isx}\, ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)(\cos sx-i\sin sx)\, ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\cos sx ds-i\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\sin sx ds$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\cos sx ds-0$$

$$\int_{0}^{\infty}\left(\frac{1-\cos as}{s^2}\right)\cos sx ds=\frac{\pi}{2}f(x)$$

Put $x=0$ *we get*

$$\int_{0}^{\infty}\left(\frac{1-\cos a s}{s^2}\right)ds=\frac{\pi}{2}f(0)$$

$$\int_0^{\infty}\left(\frac{1-\cos 2t}{\frac{4t^2}{a^2}}\right)\frac{2dt}{a}=\frac{\pi}{2}(a)$$

Put as = 2t
ads = 2dt

$f(x)$ = a – | x |
$f(0)$ = a – 0 = a

$$2a\int_0^{\infty}\left(\frac{2\sin^2 t}{4t^2}\right)dt=\frac{\pi a}{2}$$

$$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$$

This proves (i)

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty}|F(s)|^2\,ds=\int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos as}{s^2}\right]\right)^2 ds=\int_{-\infty}^{-a}0.dx+\int_{-a}^{a}(a-|x|)^2\,dx+\int_{a}^{\infty}0.dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos as}{s^2}\right)^2 ds=\int_{-a}^{a}(a-|x|)^2\,dx$$

$$\frac{4}{\pi}\int_0^{\infty}\left(\frac{1-\cos as}{s^2}\right)^2 ds=2\int_0^{a}(a-x)^2\,dx$$

$$\frac{4}{\pi}\int_0^{\infty}\left(\frac{1-\cos 2t}{4t^2/a^2}\right)^2\frac{2dt}{a}=2\left[\frac{(a-x)^3}{-3}\right]_0^a$$

Put as = 2t
ads = 2dt

$$\frac{8a^3}{16\pi}\int_0^{\infty}\left(\frac{1-\cos 2t}{t^2}\right)^2 dt=2\left[\{0\}-\left\{-\frac{a^3}{3}\right\}\right]$$

$$\frac{a^3}{2\pi}\int_0^{\infty}\left(\frac{2\sin^2 t}{t^2}\right)^2 dt=\frac{2a^3}{3}$$

$$\frac{4}{2\pi}\int_0^{\infty}\left(\frac{\sin^2 t}{t^2}\right)^2 dt=\frac{2}{3}$$

$$(i.e.)\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$$

6. Find the Fourier sine and cosine transform of $f(x)=\begin{cases}\sin x, & 0<x<a\\ 0, & x>a\end{cases}$

Sol. $F_s[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin sx\,dx$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^{a}\sin x\sin sx\,dx+\int_a^{\infty}0.\sin sx\,dx\right]$$

2sinAsinB = cos(A – B) – cos(A + B)

$$=\sqrt{\frac{2}{\pi}}\int_0^{a}\frac{1}{2}[\cos(s-1)x-\cos(s+1)x]\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)x}{s-1}-\frac{\sin(s+1)x}{s+1}\right]_0^a$$

$$=\frac{1}{\sqrt{2\pi}}\left[\left\{\frac{\sin(s-1)a}{s-1}-\frac{\sin(s+1)a}{s+1}\right\}-\{0-0\}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)a}{s-1}-\frac{\sin(s+1)a}{s+1}\right]$$

$$F_c[f(x)]=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^a \sin x\cos sx\,dx+\int_a^\infty 0.\cos sx\,dx\right]$$

2cosAsinB = sin(A + B) – sin(A – B)

$$=\sqrt{\frac{2}{\pi}}\int_0^a \frac{1}{2}[\sin(s+1)x-\sin(s-1)x]\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{-\cos(s+1)x}{s+1}+\frac{\cos(s-1)x}{s-1}\right]_0^a$$

$$=\frac{1}{\sqrt{2\pi}}\left[\left\{\frac{-\cos(s+1)a}{s+1}+\frac{\cos(s-1)a}{s-1}\right\}-\left\{\frac{-1}{s+1}+\frac{1}{s-1}\right\}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left\{\left\{\frac{(s-1)[-\cos sa\cos a+\sin sa\sin a]+(s+1)[\cos sa\cos a+\sin sa\sin a]}{(s+1)(s-1)}\right\}-\left\{\frac{-(s-1)+(s+1)}{(s+1)(s-1)}\right\}\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{2s\sin sa\sin a+2\cos sa\cos a}{s^2-1}-\frac{2}{s^2-1}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{s\sin sa\sin a+\cos sa\cos a-1}{s^2-1}\right]$$

7. Find the Fourier transform of $f(x)=\begin{cases}1-x^2, & |x|<1\\ 0, & |x|\ge 1\end{cases}$

Hence deduce that $(i)\ \int_0^\infty \frac{\sin s-s\cos s}{s^3}ds=\frac{\pi}{4}$ $\quad(ii)\ \int_0^\infty \frac{\sin s-s\cos s}{s^3}\cos\frac{s}{2}ds=\frac{3\pi}{16}$

$(iii)\ \int_0^\infty\left(\frac{\sin s-s\cos s}{s^3}\right)^2 ds=\frac{\pi}{15}$

Sol. $$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)\,e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1}0.e^{isx}\,dx+\int_{-1}^{1}(1-x^2)\,e^{isx}\,dx+\int_1^\infty 0.e^{isx}\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)\cos sx\,dx+i\,\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)\sin sx\,dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x^2)\cos sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ 0 - \frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right\} - \{0-0+0\} \right]$$

$$(i.e.) \quad F[f(x)] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

Using ***inverse Fourier transform***, we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds - i\frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx\, ds$$

$$= \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds - 0$$

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds = \frac{\pi}{4} f(x) \; -------- \; (1)$$

Put $x = 0$ *in equation* (1) *we get*

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4} f(0)$$

$$= \frac{\pi}{4}(1) = \frac{\pi}{4} \quad \textit{This proves } (i)$$

$f(x) = 1 - x^2$
$f(0) = 1 - 0 = 1$

Put $x = \frac{1}{2}$ *in equation* (1) *we get*

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2}\, ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{4}\left(1 - \frac{1}{4}\right) = \frac{\pi}{4}\left(\frac{3}{4}\right) = \frac{3\pi}{16} \quad \textit{This proves } (ii)$$

$f(x) = 1 - x^2$
$f(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

Using **Parseval's identity,** we have

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right] \right)^2 ds = \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} (1-x^2)^2\, dx + \int_1^{\infty} 0.dx$$

$$\frac{8}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = \int_{-1}^{1} (1-x^2)^2\, dx$$

$$\frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3} \right)^2 ds = 2\int_0^1 (1-x^2)^2\, dx$$

$$= 2\int_0^1 (1 + x^4 - 2x^2)\, dx$$

$$= 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[\left\{1 + \frac{1}{5} - \frac{2}{3}\right\} - \{0 + 0 - 0\}\right]$$

$$= 2\left[\frac{8}{15}\right]$$

$$\frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{15} \qquad This \ \ proves \ \ (iii)$$

MA 242303 – TRANSFORMS AND RANDOM PROCESSES

UNIT III – Z TRANSFORM

PART – A

1. Define Z – transform of the sequence {f(n)}.

Sol. If f(n) is a causal sequence (i.e.) f(n) = 0 for n < 0, then the Z – transform is called one sided (or) unilateral Z – transform of {f(n)} and is defined as

$$Z\{f(n)\} = \bar{f}(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

2. Find the Z – transform of a^n.

Sol.

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n\, z^{-n}$$

$$= \sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^{-n} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^{n}$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots\ldots\ldots\ldots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1}$$

$$= \frac{z}{z-a}$$

$$\boxed{(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots}$$

3. Find the value of Z{f(n)} where f(n) = na^n.

Sol.

$$Z\{n\,a^n\} = -z\frac{d}{dz}\left[Z(a^n)\right]$$

$$= -z\frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z\left[\frac{(z-a)(1) - z(1)}{(z-a)^2}\right]$$

$$= -z\left[\frac{-a}{(z-a)^2}\right]$$

$$= \frac{a\,z}{(z-a)^2}$$

4. Find Z{f(n)} where f(n) = n for n = 0, 1, 2, ………

Sol.

$$Z\{n\} = \sum_{n=0}^{\infty} n\, z^{-n}$$

$$= \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^{n}$$

$$= 0 + \left(\frac{a}{z}\right) + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \ldots\ldots\ldots\ldots$$

$$= \frac{a}{z}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \ldots\ldots\ldots\ldots\right]$$

$$\boxed{(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots}$$

$$= \frac{a}{z}\left(1 - \frac{a}{z}\right)^{-2} = \frac{a}{z}\left(\frac{z-a}{z}\right)^{-2}$$

$$= \frac{a}{z}\frac{z^2}{(z-a)^2} = \frac{a\,z}{(z-a)^2}$$

5. Find the Z – transform of (n + 2)

Sol. Z{n + 2} = Z(n) + Z(2)

$$= \frac{z}{(z-1)^2} + \frac{2z}{z-1} = \frac{z+2z(z-1)}{(z-1)^2} = \frac{2z^2-z}{(z-1)^2}$$

6. Find Z(1/n)

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\ldots\ldots$$

Sol. $Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \ldots\ldots\ldots\ldots\ldots$$

$$= -\log\left(1-\frac{1}{z}\right)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z}{z-1}\right)$$

7. Find the Z – transform of 3^n.

Sol. $Z\{3^n\} = \sum_{n=0}^{\infty} 3^n z^{-n}$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^{-n} = \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^{n}$$

$$= 1 + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + \ldots\ldots\ldots\ldots$$

$$= \left(1-\frac{3}{z}\right)^{-1} = \left(\frac{z-3}{z}\right)^{-1}$$

$$= \frac{z}{z-3}$$

8. Find the Z – transform of (n + 1)(n + 2)

Sol. Z{(n + 1)(n + 2)} = Z{n^2 + 3n + 2}

= Z(n^2) + 3Z(n) + Z(2)

$$= \frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$$

$$= \frac{z^2+z+3z(z-1)+2z(z-1)^2}{(z-1)^3}$$

$$= \frac{z^2+z+3z^2-3z+2z^3-4z^2+2z}{(z-1)^3}$$

$$= \frac{2z^3}{(z-1)^3}$$

9. Find the Z – transform of $\sin\frac{n\pi}{2}$

Sol. We have $Z\{r^n \sin n\theta\} = \frac{z\,r\sin\theta}{z^2-2zr\cos\theta+r^2}$

$$\therefore Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$$

10. If $Z\{f(t)\} = \bar{f}(z),\ then\ \ Z\{e^{-at} f(t)\} = \bar{f}(z\,e^{aT})$

Sol. $$Z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)\,(ze^{aT})^{-n}$$

$$= \bar{f}(ze^{aT})$$

11. Find Z[e $^{-iat}$] using Z – transform.

Sol. We have $Z\{1\} = \dfrac{z}{z-1}$

$$\therefore\ Z\left[e^{-iat}\right] = Z\left[e^{-iat}(1)\right] = \frac{z\,e^{iaT}}{z\,e^{iaT}-1}$$

12. If $Z\{f(n)\} = \bar{f}(z),\ then\ Z\{a^n f(n)\} = \bar{f}\left(\dfrac{z}{a}\right)$

Sol. $$Z\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n}$$

$$= \bar{f}\left(\frac{z}{a}\right)$$

13. Find the Z – transform of $f(n) = \begin{cases} \dfrac{a^n}{n!} & for\ \ n \ge 0 \\ 0 & otherwise \end{cases}$

Sol. $$Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{z}{a}\right)^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^{n}$$

$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots}$$

$$= 1 + \frac{1}{1!}\left(\frac{a}{z}\right) + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \ldots\ldots\ldots\ldots\ldots$$

$$= e^{a/z}$$

14. Define the unit step sequence. Write its Z – transform.

Sol. U(n) is the unit step sequence defined by

$$U(n) = \begin{cases} 1 & for\ \ n \ge 0 \\ 0 & for\ \ n < 0 \end{cases}$$

$$Z\{U(n)\} = Z(1) = \frac{z}{z-1}$$

15. State convolution theorem of Z – transform.

Sol. $If\ \ Z\{f(n)\} = \bar{f}(z)\ \ and\ \ Z\{g(n)\} = \bar{g}(z)\ \ then$

$$Z\{f(n) * g(n)\} = \bar{f}(z).\bar{g}(z)$$

16. State and prove initial value theorem in Z – transform.

Statement: If $Z\{f(n)\} = \bar{f}(z),\ then\ f(0) = \lim_{z \to \infty} \bar{f}(z)$

Proof.

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \frac{f(3)}{z^3} + ...$$

$$\lim_{z \to \infty} \bar{f}(z) = f(0)$$

17. State final value theorem in Z – transform.

Sol. $If\ Z\{f(n)\} = \bar{f}(z)\ then\ \lim_{n \to \infty} [f(n)] = \lim_{z \to 1} \{(z-1)\bar{f}(z)\}$

18. $If\ F(z) = \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)},\ find\ f(0)$

Sol. $f(0) = \lim_{z \to \infty} \bar{f}(z)$ $[\bar{f}(z) = \text{F(z)}]$

$$= \lim_{z \to \infty} \frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$$

$$= \lim_{z \to \infty} \frac{z^2}{z^3\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{4z}\right)\left(1-\frac{3}{4z}\right)}$$

$$= \lim_{z \to \infty} \frac{1}{z\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{4z}\right)\left(1-\frac{3}{4z}\right)}$$

$$= \frac{1}{\infty} = 0$$

19. Express Z{f(n + 1)} in terms of $\bar{f}(z)$

Sol. We have $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n} = \bar{f}(z)$

$$\therefore Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1)\, z^{-n}$$

$$= \sum_{m-1=0}^{\infty} f(m)\, z^{-(m-1)}$$

Put n + 1 = m
n = m – 1

$$= z \sum_{m=1}^{\infty} f(m)\, z^{-m}$$

$$= z\left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - f(0)\right]$$

$$(i.e.)\ Z\{f(n+1)\} = z[\bar{f}(z) - f(0)]$$

20. Form a difference equation by eliminating the arbitrary constant A from $y_n = A.3^n$

Sol. $y_n = A.3^n$

$$y_{n+1} = A.3^{n+1}$$

$$= 3A.3^n = 3y_n$$

$$(i.e.)\ y_{n+1} - 3y_n = 0$$

21. Form a difference equation by eliminating arbitrary constant from $U_n = a.2^{n+1}$

Sol.

$$U_n = a.2^{n+1}$$

$$U_{n+1} = a.2^{n+2}$$

$$= 2a.2^{n+1} = 2U_n$$

$$(i.e.)\ U_{n+1} - 2U_n = 0$$

22. Form the difference equation from $y_n = a + b.3^n$

Sol. Given $y_n = a + b.3^n$ ------------- (1)

$$y_{n+1} = a + b.3^{n+1}$$

$$= a + 3b.3^n \ ------- \ (2)$$

$$y_{n+2} = a + b.3^{n+2}$$

$$= a + 9b.3^n \ ------- \ (3)$$

Eliminating a and b from equations (1), (2) and (3), we have

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 3 \\ y_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$y_n(6) - y_{n+1}(8) + y_{n+2}(2) = 0$$

$$(i.e.)\ y_{n+2} - 4y_{n+1} + 3y_n = 0$$

23. Form the difference equation by eliminating the constants A and B from $y_n = A(-2)^n + B.3^n$

Sol. Given $y_n = A(-2)^n + B.3^n$ ------------- (1)

$$y_{n+1} = A(-2)^{n+1} + B.3^{n+1}$$

$$= -2A(-2)^n + 3B.3^n \ ------- \ (2)$$

$$y_{n+2} = A(-2)^{n+2} + B.3^{n+2}$$

$$= 4A(-2)^n + 9B.3^n \ ------- \ (3)$$

Eliminating A and B from equations (1), (2) and (3), we have

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & -2 & 3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$$y_n(-30) - y_{n+1}(5) + y_{n+2}(5) = 0$$

$$(i.e.)\ y_{n+2} - y_{n+1} - 6y_n = 0$$

24. Find the difference equation generated by $y_n = a\,n + b.2^n$

Sol. Given $y_n = a\,n + b.2^n$ ------------- (1)

$$y_{n+1} = a(n+1) + b.2^{n+1}$$

$$= a(n+1) + 2b.2^n \ ------- \ (2)$$

$$y_{n+2} = a(n+2) + b.2^{n+2}$$

$$= a(n+2) + 4b.2^n \ ------- \ (3)$$

Eliminating a and b from equations (1), (2) and (3), we have

$$\begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 4 \end{vmatrix} = 0$$

$$y_n[4(n+1)-2(n+2)]-y_{n+1}[4n-(n+2)]+y_{n+2}[2n-(n+1)]=0$$
$$y_n(2n)-y_{n+1}(3n-2)+y_{n+2}(n-1)=0$$
$$(i.e.)\ \ (n-1)y_{n+2}-(3n-2)y_{n+1}+2ny_n=0$$

25. Evaluate $Z^{-1}\left[\dfrac{z}{z^2+7z+10}\right]$

Sol. $Let\ \bar{f}(z)=\dfrac{z}{(z+2)(z+5)}$

$$\frac{\bar{f}(z)}{z}=\frac{1}{(z+2)(z+5)}=\frac{A}{z+2}+\frac{B}{z+5}$$
$$1=A(z+5)+B(z+2)$$
$$Put\ \ z=-2,\ we\ get\quad 1=A(3)+0$$
$$\Rightarrow\ A=\frac{1}{3}$$
$$Put\ \ z=-5,\ we\ get\quad 1=0+B(-3)$$
$$\Rightarrow\ B=-\frac{1}{3}$$
$$\frac{\bar{f}(z)}{z}=\frac{1/3}{z+2}+\frac{-1/3}{z+5}$$
$$\bar{f}(z)=\frac{1}{3}\frac{z}{z+2}-\frac{1}{3}\frac{z}{z+5}$$
$$\therefore Z^{-1}\left[\bar{f}(z)\right]=\frac{1}{3}Z^{-1}\left[\frac{z}{z+2}\right]-\frac{1}{3}Z^{-1}\left[\frac{z}{z+5}\right]$$
$$=\frac{1}{3}(-2)^n-\frac{1}{3}(-5)^n$$

26. Does the Z – transform of n! exist? Justify your answer.

Sol. $Z\{n!\}=\sum_{n=0}^{\infty}n!z^{-n}$

$$=1+\frac{1!}{z}+\frac{2!}{z^2}+\frac{3!}{z^3}+.............$$

Thus the Z – transform of n! does not exist.

27. What advantage is gained when Z – transform is used to solve difference equation?

Sol. The role played by the Z – transform in the solution of difference equations corresponds to that played by the Laplace transform in the solution of differential equations.

PART – B

1. Find the Z – transform of the sequences f_n = (n + 1)(n + 2) and g_n = n(n – 1)

Sol. $Z\{f(n)\} = Z\{(n+1)(n+2)\}$

$$= Z\{n^2 + 3n + 2\}$$

$$= Z\{n^2\} + 3Z\{n\} + Z(2)$$

$$= \frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + \frac{2z}{z-1}$$

$$= \frac{z^2 + z + 3z(z-1) + 2z(z-1)^2}{(z-1)^3}$$

$$= \frac{z^2 + z + 3z^2 - 3z + 2z^3 - 4z^2 + 2z}{(z-1)^3}$$

$$= \frac{2z^3}{(z-1)^3}$$

$Z\{g(n)\} = Z\{n(n-1)\}$

$$= Z\{n^2 - n\}$$

$$= Z\{n^2\} - Z\{n\}$$

$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z^2 + z - z(z-1)}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}$$

2. Find the Z – transform $(i)\ \{a^n\} \quad (ii)\ \{n a^n\}$

Sol. (i) $Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^{n}$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \ldots\ldots\ldots\ldots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1}$$

$$= \frac{z}{z-a}$$

(ii) $Z\{n a^n\} = -z\frac{d}{dz}\left[Z(a^n)\right]$

$$= -z\frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z\left[\frac{(z-a)(1) - z(1)}{(z-a)^2}\right]$$

$$= -z\left[\frac{-a}{(z-a)^2}\right]$$

$$= \frac{a z}{(z-a)^2}$$

3. Find the Z – transform $\left\{\frac{1}{n}\right\}, \left\{\cos\frac{n\pi}{2}\right\}, \left\{\sin\frac{n\pi}{2}\right\}, \{a^n\cos n\pi\}$ *and* $\left\{a^n\cos\frac{n\pi}{2}\right\}$

Sol. (i) $Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty}\frac{1}{n}z^{-n}$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + ...$$

$$= -\log\left(1-\frac{1}{z}\right)$$

$$= -\log\left(\frac{z-1}{z}\right)$$

$$= \log\left(\frac{z}{z-1}\right)$$

(*ii*) *We know that*

$$Z\{a^n\} = \frac{z}{z-a}$$

put $a = re^{i\theta}$, *we get*

$$Z\{(re^{i\theta})^n\} = \frac{z}{z-re^{i\theta}}$$

$$Z\{r^n e^{in\theta}\} = \frac{z}{z-re^{i\theta}}$$

$$Z\{r^n(\cos n\theta + i\sin n\theta)\} = \frac{z}{z-r(\cos\theta+i\sin\theta)}$$

$$Z\{r^n\cos n\theta + i\,r^n\sin n\theta\} = \frac{z}{(z-r\cos\theta)-i\,r\sin\theta}$$

$$= \frac{z[(z-r\cos\theta)+i\,r\sin\theta]}{[(z-r\cos\theta)-i\,r\sin\theta][(z-r\cos\theta)+i\,r\sin\theta]}$$

$$= \frac{z(z-r\cos\theta)+i\,z\,r\sin\theta}{(z-r\cos\theta)^2+r^2\sin^2\theta}$$

$$= \frac{z(z-r\cos\theta)+i\,z\,r\sin\theta}{z^2-2zr\cos\theta+r^2}$$

Equating R.P and I.P, we get

$$Z\{r^n\cos n\theta\} = \frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2} \quad and \quad Z\{r^n\sin n\theta\} = \frac{z\,r\sin\theta}{z^2-2zr\cos\theta+r^2}$$

$$\Rightarrow Z\left\{a^n\cos\frac{n\pi}{2}\right\} = \frac{z^2}{z^2+a^2}$$

$$Z\left\{\cos\frac{n\pi}{2}\right\} = \frac{z^2}{z^2+1} \quad and \quad Z\left\{\sin\frac{n\pi}{2}\right\} = \frac{z}{z^2+1}$$

Now, $Z\{a^n\cos n\pi\} = Z\{a^n(-1)^n\} = Z\{(-a)^n\} = \frac{z}{z+a}$

4. Find the Z – transform $(i)\ n\cos n\theta \quad (ii)\ n\,a^n\sin n\theta$

Sol. $Z\{n\cos n\theta\} = -z\frac{d}{dz}\left[Z(\cos n\theta)\right]$

$$= -z\frac{d}{dz}\left[\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]$$

$$= -z\left[\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{(2z^3-5z^2\cos\theta+2z\cos^2\theta+2z-\cos\theta)-(2z^3-4z^2\cos\theta+2z\cos^2\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{-z^2\cos\theta+2z-\cos\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= z\left[\frac{z^2\cos\theta-2z+\cos\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$Z\{n\,a^n\sin n\theta\} = -z\frac{d}{dz}\left[Z(a^n\sin n\theta)\right]$$

$$= -z\frac{d}{dz}\left[\frac{za\sin\theta}{z^2-2za\cos\theta+a^2}\right]$$

$$= -z\left[\frac{(z^2-2za\cos\theta+a^2)(a\sin\theta)-za\sin\theta(2z-2a\cos\theta)}{(z^2-2za\cos\theta+a^2)^2}\right]$$

$$= -z\left[\frac{z^2a\sin\theta-2za^2\sin\theta\cos\theta+a^3\sin\theta-2z^2a\sin\theta+2za^2\sin\theta\cos\theta}{(z^2-2za\cos\theta+a^2)^2}\right]$$

$$= -z\left[\frac{-z^2a\sin\theta+a^3\sin\theta}{(z^2-2za\cos\theta+a^2)^2}\right]$$

$$= \frac{(z^2-a^2)za\sin\theta}{(z^2-2za\cos\theta+a^2)^2}$$

5. Find the Z – transform $(i)\ \sin^2\left(\frac{n\pi}{4}\right)\quad (ii)\ \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

Sol. (i) $Z\left[\sin^2\left(\frac{n\pi}{4}\right)\right] = Z\left[\frac{1}{2}\left(1-\cos\frac{2n\pi}{4}\right)\right]$

$$= \frac{1}{2}\left[Z(1)-Z\left(\cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-1}-\frac{z^2}{z^2+1}\right]$$

(ii) $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = Z\left[\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right]$

$$= Z\left[\cos\frac{n\pi}{2}.\frac{1}{\sqrt{2}}-\sin\frac{n\pi}{2}.\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}\left[Z\left(\cos\frac{n\pi}{2}\right)-Z\left(\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1}-\frac{z}{z^2+1}\right]$$

$$= \frac{1}{\sqrt{2}}\frac{z(z-1)}{z^2+1}$$

6. Find Z{f(n)} if $f(n)=\frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$

Sol. $$Z\{f(n)\}=Z\left[\frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\right]$$

$$=\frac{1}{\sqrt{5}}\left[Z\left\{\left(\frac{\sqrt{5}+1}{2}\right)^n\right\}-Z\left\{\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{z}{z-\{(\sqrt{5}+1)/2\}}-\frac{z}{z-\{(1-\sqrt{5})/2\}}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{2z}{2z-(\sqrt{5}+1)}-\frac{2z}{2z-(1-\sqrt{5})}\right]$$

$$=\frac{2}{\sqrt{5}}\left\{\frac{z[2z-(1-\sqrt{5})]-z[2z-(\sqrt{5}+1)]}{[2z-(\sqrt{5}+1)][2z-(1-\sqrt{5})]}\right\}$$

$$=\frac{2}{\sqrt{5}}\left\{\frac{2z^2-z+z\sqrt{5}-2z^2+z\sqrt{5}+z}{4z^2-2z(1-\sqrt{5})-2z(\sqrt{5}+1)+(1+\sqrt{5})(1-\sqrt{5})}\right\}$$

$$=\frac{2}{\sqrt{5}}\left\{\frac{2z\sqrt{5}}{4z^2-2z+2z\sqrt{5}-2z\sqrt{5}-2z+(1-5)}\right\}$$

$$=\frac{2}{\sqrt{5}}\left\{\frac{2z\sqrt{5}}{4z^2-4z-4}\right\}$$

$$=\frac{z}{z^2-z-1}$$

7. Find the Z – transform of (i) $\frac{2n+3}{(n+1)(n+2)}$ **(ii)** $\frac{1}{n(n-1)}$

Sol. (i) $$\frac{2n+3}{(n+1)(n+2)}=\frac{A}{n+1}+\frac{B}{n+2}$$

$$2n+3=A(n+2)+B(n+1)$$

$$Put\ \ n=-1,\ we\ get\quad 1=A(1)+0$$

$$\Rightarrow\ A=1$$

$$Put\ \ n=-2,\ we\ get\quad -1=0+B(-1)$$

$$\Rightarrow\ B=1$$

$$\frac{2n+3}{(n+1)(n+2)}=\frac{1}{n+1}+\frac{1}{n+2}$$

$$Z\left[\frac{2n+3}{(n+1)(n+2)}\right]=Z\left[\frac{1}{n+1}\right]+Z\left[\frac{1}{n+2}\right]\ ------\ (1)$$

$$Z\left[\frac{1}{n+1}\right]=\sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n}$$

$$=1+\frac{1}{2z}+\frac{1}{3z^2}+\frac{1}{4z^3}+\ldots\ldots\ldots\ldots.$$

$$=z\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+\ldots\ldots\ldots\ldots.\right]$$

$$=z\left[-\log\left(1-\frac{1}{z}\right)\right]$$

$$\boxed{-\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots\ldots.}$$

$$= -z\log\left(\frac{z-1}{z}\right)$$
$$= z\log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty}\frac{1}{n+2}z^{-n}$$
$$= \frac{1}{2}+\frac{1}{3z}+\frac{1}{4z^2}+.............$$
$$= z^2\left[\frac{1}{2z^2}+\frac{1}{3z^3}+\frac{1}{4z^4}+.............\right]$$
$$= z^2\left[-\log\left(1-\frac{1}{z}\right)-\frac{1}{z}\right]$$
$$= -z^2\log\left(\frac{z-1}{z}\right)-z$$
$$= z^2\log\left(\frac{z}{z-1}\right)-z$$

Equation (1) becomes

$$Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = z\log\left(\frac{z}{z-1}\right)+z^2\log\left(\frac{z}{z-1}\right)-z$$
$$= z(1+z)\log\left(\frac{z}{z-1}\right)-z$$

(ii) $$\frac{1}{n(n-1)} = \frac{A}{n}+\frac{B}{n-1}$$

$$1 = A(n-1)+B(n)$$

$Put\ \ n=0,\ we\ \ get \qquad 1 = A(-1)+0$

$$\Rightarrow\ A=-1$$

$Put\ \ n=1,\ we\ \ get \qquad 1 = 0+B(1)$

$$\Rightarrow\ B=1$$

$$\frac{1}{n(n-1)} = \frac{-1}{n}+\frac{1}{n-1}$$

$$Z\left[\frac{1}{n(n-1)}\right] = -Z\left[\frac{1}{n}\right]+Z\left[\frac{1}{n-1}\right] \text{ ------ } (1)$$

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty}\frac{1}{n}z^{-n}$$
$$= \frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+.............$$
$$= -\log\left(1-\frac{1}{z}\right)$$
$$= -\log\left(\frac{z-1}{z}\right)$$
$$= \log\left(\frac{z}{z-1}\right)$$

$$Z\left[\frac{1}{n-1}\right]=\sum_{n=2}^{\infty}\frac{1}{n-1}z^{-n}$$
$$=\frac{1}{z^2}+\frac{1}{2z^3}+\frac{1}{3z^4}+.............$$
$$=\frac{1}{z}\left[\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{3z^3}+.............\right]$$
$$=\frac{1}{z}\left[-\log\left(1-\frac{1}{z}\right)\right]$$
$$=-\frac{1}{z}\log\left(\frac{z-1}{z}\right)$$
$$=\frac{1}{z}\log\left(\frac{z}{z-1}\right)$$

Equation (1) becomes

$$Z\left[\frac{1}{n(n-1)}\right]=-\log\left(\frac{z}{z-1}\right)+\frac{1}{z}\log\left(\frac{z}{z-1}\right)$$
$$=\left(-1+\frac{1}{z}\right)\log\left(\frac{z}{z-1}\right)$$
$$=\left(\frac{1-z}{z}\right)\log\left(\frac{z}{z-1}\right)\quad (or)\quad \left(\frac{z-1}{z}\right)\log\left(\frac{z-1}{z}\right)$$

8. State and prove the second shifting theorem in Z – transform.

Statement: $If\ \ Z\{f(n)\}=\bar{f}(z)\ \ then\ \ Z\{f(n+1)\}=z[\bar{f}(z)-f(0)]$

Proof. We have

$$Z\{f(n)\}=\sum_{n=0}^{\infty}f(n)z^{-n}=\bar{f}(z)$$
$$\therefore Z\{f(n+1)\}=\sum_{n=0}^{\infty}f(n+1)z^{-n}$$
$$=\sum_{m-1=0}^{\infty}f(m)z^{-(m-1)}$$
$$=z\sum_{m=1}^{\infty}f(m)z^{-m}$$
$$=z\left[\sum_{m=0}^{\infty}f(m)z^{-m}-f(0)\right]$$

Put n + 1 = m
n = m – 1

$$(i.e.)\ Z\{f(n+1)\}=z[\bar{f}(z)-f(0)]$$

Note:

$$Similarly,\ \ Z\{f(n+2)\}=\sum_{m-2=0}^{\infty}f(m)z^{-(m-2)}=z^2\sum_{m=2}^{\infty}f(m)z^{-m}$$
$$=z^2\left[\sum_{m=0}^{\infty}f(m)z^{-m}-f(0)-f(1)z^{-1}\right]$$
$$=z^2[\bar{f}(z)-f(0)-f(1)z^{-1}]$$
$$=z^2\bar{f}(z)-z^2f(0)-zf(1)$$
$$Z\{f(n+3)\}=z^3\bar{f}(z)-z^3f(0)-z^2f(1)-zf(2)\ \ and\ \ so\ \ on.$$

In general,

$$Z\{f(n+k)\}=z^k[\bar{f}(z)-f(0)-f(1)z^{-1}-f(2)z^{-2}-...........-f(k-1)z^{-(k-1)}]$$

9. State and prove final value theorem in Z – transform.

Statement: *If* $Z\{f(n)\} = \bar{f}(z)$ *then* $\lim_{n\to\infty} [f(n)] = \lim_{z\to 1} \{(z-1)\bar{f}(z)\}$

Proof. By definition,

$$Z\{f(n+1) - f(n)\} = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$[z\bar{f}(z) - z f(0)] - \bar{f}(z) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

$$(i.e.)\ \ (z-1)\bar{f}(z) - z f(0) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]z^{-n}$$

Taking limit as z→1 on both sides, we get

$$\lim_{z\to 1} [(z-1)\bar{f}(z)] - f(0) = \sum_{n=0}^{\infty}[f(n+1) - f(n)]$$

$$\begin{aligned}\lim_{z\to 1} [(z-1)\bar{f}(z)] - f(0) &= \lim_{n\to\infty} [f(1) - f(0) \\ &\qquad + f(2) - f(1) \\ &\qquad + f(3) - f(2) \\ &\qquad + \ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad + f(n+1) - f(n)] \\ &= \lim_{n\to\infty} [f(n+1) - f(0)] \\ &= \lim_{n\to\infty} [f(n)] - f(0)\end{aligned}$$

$$\text{(i.e.)}\ \ \lim_{z\to 1} [(z-1)\bar{f}(z)] = \lim_{n\to\infty} [f(n)]$$

10. State and prove convolution theorem in Z – transform.

Statement: *If* $Z\{f(n)\} = \bar{f}(z)$ *and* $Z\{g(n)\} = \bar{g}(z)$ *then* $Z\{f(n) * g(n)\} = \bar{f}(z).\bar{g}(z)$

Proof. We have

$$\bar{f}(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}\ ,\quad \bar{g}(z) = \sum_{n=0}^{\infty} g(n)\, z^{-n}$$

$$\begin{aligned}\bar{f}(z).\bar{g}(z) = [f(0) + f(1)z^{-1} + f(2)z^{-2} + f(3)z^{-3} + \ldots\ldots\ldots\ldots\infty] \times \\ [g(0) + g(1)z^{-1} + g(2)z^{-2} + g(3)z^{-3} + \ldots\ldots\ldots\ldots\infty]\end{aligned}$$

$$\begin{aligned}&= \sum_{n=0}^{\infty}[f(0)\,g(n) + f(1)\,g(n-1) + f(2)\,g(n-2) + \ldots\ldots\ldots + f(n)\,g(0)]\,z^{-n} \\ &= Z[f(0)\,g(n) + f(1)\,g(n-1) + f(2)\,g(n-2) + \ldots\ldots\ldots + f(n)\,g(0)] \\ &= Z\{f(n) * g(n)\}\end{aligned}$$

11. Find the inverse Z – transform of $\frac{10z}{z^2-3z+2}$

Sol. $Let\ \bar{f}(z)=\frac{10z}{(z-1)(z-2)}$

$$\frac{\bar{f}(z)}{z}=\frac{10}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$$

$$10=A(z-2)+B(z-1)$$

$$Put\ \ z=1,\ we\ get\quad 10=A(-1)+0$$

$$\Rightarrow\ A=-10$$

$$Put\ \ z=2,\ we\ get\quad 10=0+B(1)$$

$$\Rightarrow\ B=10$$

$$\frac{\bar{f}(z)}{z}=\frac{-10}{z-1}+\frac{10}{z-2}$$

$$\bar{f}(z)=\frac{-10z}{z-1}+\frac{10z}{z-2}$$

$$\therefore Z^{-1}\left[\bar{f}(z)\right]=-10Z^{-1}\left[\frac{z}{z-1}\right]+10Z^{-1}\left[\frac{z}{z-2}\right]$$

$$=-10(1)^n+10(2)^n$$

$$=-10+10.2^n$$

12. Find $Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$

Sol. $\frac{z}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2}$

$$z=A(z-2)+B(z-1)$$

$$Put\ \ z=1,\ we\ get\quad 1=A(-1)+0$$

$$\Rightarrow\ A=-1$$

$$Put\ \ z=2,\ we\ get\quad 2=0+B(1)$$

$$\Rightarrow\ B=2$$

$$\frac{z}{(z-1)(z-2)}=\frac{-1}{z-1}+\frac{2}{z-2}$$

$$\therefore Z^{-1}\left[\frac{z}{(z-1)(z-2)}\right]=-Z^{-1}\left[\frac{1}{z-1}\right]+2Z^{-1}\left[\frac{1}{z-2}\right]$$

$$=-(1)^{n-1}+2(2)^{n-1}$$

$$=-1+2\,\frac{2^n}{2}$$

$$=-1+2^n$$

13. Find $Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right]$ **using partial fraction method.**

Sol. $Let\ \bar{f}(z)=\frac{z^3}{(z-1)^2(z-2)}$

$$\frac{\bar{f}(z)}{z}=\frac{z^2}{(z-1)^2(z-2)}=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z-2}$$

$$z^2=A(z-1)(z-2)+B(z-2)+C(z-1)^2$$

$$Put\ \ z=1,\ we\ \ get\qquad 1=0+B(-1)+0$$
$$\Rightarrow\ B=-1$$
$$Put\ \ z=2,\ we\ \ get\qquad 4=0+0+C(1)$$
$$\Rightarrow\ C=4$$
$$Coeff.\,of\ \ z^2,\qquad 1=A+C$$
$$1=A+4$$
$$\Rightarrow A=-3$$
$$\frac{\bar f(z)}{z}=\frac{-3}{z-1}+\frac{-1}{(z-1)^2}+\frac{4}{z-2}$$
$$\bar f(z)=-\frac{3z}{z-1}-\frac{z}{(z-1)^2}+\frac{4z}{z-2}$$
$$\therefore\ Z^{-1}\{\bar f(z)\}=-3Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z}{(z-1)^2}\right]+4Z^{-1}\left[\frac{z}{z-2}\right]$$
$$=-3-n+4.2^n$$

14. Find $Z^{-1}\left[\frac{z^2}{(z+2)(z^2+4)}\right]$ by the method of partial fractions.

Sol. $Let\ \ \bar f(z)=\frac{z^2}{(z+2)(z^2+4)}$

$$\frac{\bar f(z)}{z}=\frac{z}{(z+2)(z^2+4)}=\frac{A}{z+2}+\frac{Bz+C}{z^2+4}$$
$$z=A(z^2+4)+(Bz+C)(z+2)$$
$$Put\ \ z=-2,\ we\ \ get\qquad -2=A(4+4)+0$$
$$-2=8A$$
$$\Rightarrow\ A=-\frac{1}{4}$$
$$Coeff.\,of\ \ z^2,\qquad 0=A+B$$
$$0=-\frac{1}{4}+B$$
$$\Rightarrow B=\frac{1}{4}$$

(Bz + C)(z + 2)
Bz^2 + 2Bz +Cz + 2C

$$Coeff.\,of\ \ z,\qquad 1=2B+C$$
$$1=\frac{2}{4}+C$$
$$\Rightarrow C=1-\frac{1}{2}=\frac{1}{2}$$
$$\frac{\bar f(z)}{z}=\frac{-1/4}{z+2}+\frac{1/4z+1/2}{z^2+4}$$
$$\bar f(z)=-\frac{1}{4}\frac{z}{z+2}+\frac{1}{4}\frac{z^2}{z^2+4}+\frac{1}{2}\frac{z}{z^2+4}$$
$$\therefore Z^{-1}\{\bar f(z)\}=-\frac{1}{4}Z^{-1}\left[\frac{z}{z+2}\right]+\frac{1}{4}Z^{-1}\left[\frac{z^2}{z^2+4}\right]+\frac{1}{4}Z^{-1}\left[\frac{2z}{z^2+4}\right]$$
$$=-\frac{1}{4}(-2)^n+\frac{1}{4}2^n\cos\frac{n\pi}{2}+\frac{1}{4}2^n\sin\frac{n\pi}{2}$$

15. Find the inverse Z – transform of $\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$

Sol. $Let\ \bar{f}(z)=\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$

$$\frac{\bar{f}(z)}{z}=\frac{z^2+3}{(z-1)^2(z^2+1)}=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{Cz+D}{z^2+1}$$

$$z^2+3=A(z-1)(z^2+1)+B(z^2+1)+(Cz+D)(z-1)^2$$

$Put\ z=1,\ we\ get\quad 4=0+B(2)+0$

$\Rightarrow\ B=2$

A(z – 1)(z^2 + 1)
A(z^3 – z^2 + z – 1)

(Cz + D)(z – 1)^2
(Cz + D)(z^2 – 2z + 1)
Cz^3 – 2Cz^2 + Dz^2 + Cz – 2Dz + D

$Coeff.\ of\ z^3,\quad 0=A+C\ \ ------ (1)$

$Coeff.\ of\ z^2,\quad 1=-A+B-2C+D$

$1=-A+2-2C+D$

$A+2C-D=1\ \ ------ (2)$

$Coeff.\ of\ z,\quad 0=A+C-2D$

$0=0-2D$ [using (1)]

$\Rightarrow D=0$

(2) $\Rightarrow$ A + 2C = 1 ---------- (3)

(3) – (1) $\Rightarrow$ C = 1

(1) $\Rightarrow$ A = –1

$$\frac{\bar{f}(z)}{z}=\frac{-1}{z-1}+\frac{2}{(z-1)^2}+\frac{z+0}{z^2+1}$$

$$\bar{f}(z)=\frac{-z}{z-1}+\frac{2z}{(z-1)^2}+\frac{z^2}{z^2+1}$$

$$\therefore Z^{-1}\{\bar{f}(z)\}=-Z^{-1}\left[\frac{z}{z-1}\right]+2Z^{-1}\left[\frac{z}{(z-1)^2}\right]+Z^{-1}\left[\frac{z^2}{z^2+1}\right]$$

$$=-1+2n+\cos\frac{n\pi}{2}$$

16. Find the inverse Z – transform of $\dfrac{z^3-20z}{(z-2)^3(z-4)}$

Sol. Let $Z^{-1}\{\bar{f}(z)\}=f(n)=$ sum of the residues of $\left\{\dfrac{z^3-20z}{(z-2)^3(z-4)}.z^{n-1}\right\}$ at its poles.

(i.e.) $f(n)=$ sum of the residues of $\left\{\dfrac{z^{n+2}-20z^n}{(z-2)^3(z-4)}\right\}$ at its poles.

$Poles\ of\ \ \bar{f}(z).z^{n-1}\ \ are$

$(z-2)^3(z-4)=0$

$\Rightarrow\ z=2,4$

z = 2 is the pole of order 3

and z = 4 is the simple pole.

$$\text{Re}\,s(z=2)=\frac{1}{2!}\lim_{z\to 2}\left[\frac{d^2}{dz^2}\left\{(z-2)^3\frac{z^{n+2}-20z^n}{(z-2)^3(z-4)}\right\}\right]$$

$$=\frac{1}{2}\lim_{z\to 2}\left[\frac{d^2}{dz^2}\left\{\frac{z^{n+2}-20z^n}{z-4}\right\}\right]$$

$$=\frac{1}{2}\lim_{z\to 2}\left[\frac{d}{dz}\left\{\frac{(z-4)[(n+2)z^{n+1}-20n\,z^{n-1}]-(z^{n+2}-20\,z^{n})(1)}{(z-4)^2}\right\}\right]$$

$$=\frac{1}{2}\lim_{z\to 2}\left[\frac{d}{dz}\left\{\frac{z^{n+2}(n+1)+z^{n}(-20n+20)-4(n+2)\,z^{n+1}+80n\,z^{n-1}}{(z-4)^2}\right\}\right]$$

$$=\frac{1}{2}\lim_{z\to 2}\left[\frac{\begin{array}{r}(z-4)^2\{(n+1)(n+2)z^{n+1}+(-20n+20)\,n\,z^{n-1}-4(n+2)(n+1)z^{n}+80n\,(n-1)\,z^{n-2}\}\\ -\{z^{n+2}(n+1)+z^{n}(-20n+20)-4(n+2)\,z^{n+1}+80n\,z^{n-1}\}.2(z-4)\end{array}}{(z-4)^4}\right]$$

$$=\frac{1}{2}\left[\frac{\begin{array}{r}(-2)^2\{(n+1)(n+2)2^{n}.2+(-10n+10)\,n\,2^{n}-4(n+2)(n+1)2^{n}+20n\,(n-1)\,2^{n}\}\\ -\{2^{n}.4(n+1)+2^{n}(-20n+20)-4(n+2)\,2^{n}.2+40n\,2^{n}\}.2(-2)\end{array}}{(-2)^4}\right]$$

$$=\frac{4.2^n}{2}\left[\frac{-2(n^2+3n+2)+10n^2-10n+4n+4-20n+20-8n-16+40n}{16}\right]$$

$$=\frac{4.2^n}{2}\left[\frac{8n^2+4}{16}\right]$$

$$=\frac{2^n}{2}\,(2n^2+1)$$

$$\operatorname{Re}s(z=4)=\lim_{z\to 4}\;(z-4)\frac{z^{n+2}-20\,z^{n}}{(z-2)^3(z-4)}$$

$$=\lim_{z\to 4}\;\frac{z^{n+2}-20\,z^{n}}{(z-2)^3}$$

$$=\frac{4^{n+2}-20\,4^{n}}{(2)^3}$$

$$=\frac{4^n}{8}(16-20)$$

$$=-\frac{4^n}{2}$$

$$\therefore\; f(n)=\operatorname{Re}s(z=2)+\operatorname{Re}s(z=4)$$

$$=\frac{2^n}{2}(2n^2+1)-\frac{4^n}{2}$$

ALITER

16. Find the inverse Z – transform of $\dfrac{z^3-20z}{(z-2)^3(z-4)}$

Sol. *Let* $\bar{f}(z)=\dfrac{z^3-20z}{(z-2)^3(z-4)}$

$$\frac{\bar{f}(z)}{z}=\frac{z^2-20}{(z-2)^3(z-4)}=\frac{A}{z-2}+\frac{B}{(z-2)^2}+\frac{C}{(z-2)^3}+\frac{D}{z-4}$$

$$z^2-20=A(z-2)^2(z-4)+B(z-2)(z-4)+C(z-4)+D(z-2)^3$$

Put $z=2$, *we get* $\quad -16=0+0+C(-2)+0$

$$\Rightarrow C=8$$

Put $z=4$, *we get* $\quad -4=0+0+0+D(2)^3$

$$\Rightarrow D=-\frac{4}{8}=-\frac{1}{2}$$

$$\textit{Coeff. of } z^3, \quad 0 = A + D$$

$$0 = A - \frac{1}{2}$$

$$\Rightarrow A = \frac{1}{2}$$

$$\textit{Coeff. of } z^2, \quad 1 = -8A + B - 6D$$

$$1 = -\frac{8}{2} + B + \frac{6}{2}$$

$$1 = -1 + B \;\; \Rightarrow B = 2$$

A(z – 2)^2(z – 4)
A(z^2 – 4z + 4)(z – 4)
A(z^3 – 8z^2 + 20z – 16)

D(z – 2)^3
D(z^3 – 6z^2 + 12z – 8)

$$\frac{\bar{f}(z)}{z} = \frac{1/2}{z-2} + \frac{2}{(z-2)^2} + \frac{8}{(z-2)^3} + \frac{-1/2}{z-4}$$

$$\bar{f}(z) = \frac{1}{2}\frac{z}{z-2} + \frac{2z}{(z-2)^2} + \frac{8z}{(z-2)^3} - \frac{1}{2}\frac{z}{z-4}$$

$$= \frac{1}{2}\frac{z}{z-2} + \frac{2z(z-2)+8z}{(z-2)^3} - \frac{1}{2}\frac{z}{z-4}$$

$$= \frac{1}{2}\frac{z}{z-2} + \frac{2z^2+4z}{(z-2)^3} - \frac{1}{2}\frac{z}{z-4}$$

$$Z^{-1}\{\bar{f}(z)\} = \frac{1}{2}Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-4}\right]$$

$$= \frac{2^n}{2} + 2^n n^2 - \frac{4^n}{2}$$

$$Z^{-1}\left[\frac{az^2 + a^2 z}{(z-a)^3}\right] = a^n n^2$$

17. Find $Z^{-1}\left[\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$ using residue method.

Sol. Let $Z^{-1}\{\bar{f}(z)\} = f(n) =$ sum of the residues of $\left\{\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}.z^{n-1}\right\}$ at its poles.

(i.e.) $f(n) =$ sum of the residues of $\left\{\dfrac{z^n(z^2-z+2)}{(z+1)(z-1)^2}\right\}$ at its poles.

$$\textit{Poles of } \bar{f}(z).z^{n-1} \textit{ are}$$

$$(z+1)(z-1)^2 = 0$$

$$\Rightarrow z = -1, 1$$

z = –1 is the simple pole
and z = 1 is the pole of order 2.

$$\operatorname{Re}s(z=-1) = \lim_{z \to -1} (z+1)\frac{z^n(z^2-z+2)}{(z+1)(z-1)^2}$$

$$= \lim_{z \to -1} \frac{z^n(z^2-z+2)}{(z-1)^2}$$

$$= \frac{(-1)^n(1+1+2)}{4}$$

$$= (-1)^n$$

$$\operatorname{Re}s(z=1) = \frac{1}{1!}\lim_{z \to 1}\left[\frac{d}{dz}\left\{(z-1)^2\frac{z^n(z^2-z+2)}{(z+1)(z-1)^2}\right\}\right]$$

$$= \lim_{z \to 1}\left[\frac{d}{dz}\left\{\frac{z^n(z^2-z+2)}{(z+1)}\right\}\right]$$

$$= \lim_{z \to 1} \left[\frac{(z+1)\{z^n(2z-1)+(z^2-z+2).n\,z^{n-1}\} - z^n(z^2-z+2)(1)}{(z+1)^2} \right]$$

$$= \left[\frac{(2)\{1+(2).n\}-(2)(1)}{(2)^2} \right]$$

$$= \frac{2+4n-2}{4}$$

$$= n$$

$$\therefore\ f(n) = \operatorname{Re}s(z=-1) + \operatorname{Re}s(z=1)$$

$$= (-1)^n + n$$

18. Find the inverse Z – transform of $\frac{z(z+1)}{(z-1)^3}$ by residue method.

Sol. Let $Z^{-1}\{\bar{f}(z)\} = f(n) =$ sum of the residues of $\left\{ \frac{z(z+1)}{(z-1)^3}.z^{n-1} \right\}$ at its poles.

(i.e.) $f(n) =$ sum of the residues of $\left\{ \frac{z^n(z+1)}{(z-1)^3} \right\}$ at its poles.

Poles of $\bar{f}(z).z^{n-1}$ *are*

$$(z-1)^3 = 0$$

$$\Rightarrow z = 1$$

z = 1 is the pole of order 3.

$$\operatorname{Re}s(z=1) = \frac{1}{2!} \lim_{z \to 1} \left[\frac{d^2}{dz^2} \left\{ (z-1)^3 \frac{z^n(z+1)}{(z-1)^3} \right\} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} \left[\frac{d^2}{dz^2} \{z^n(z+1)\} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} \left[\frac{d}{dz} \{z^n(1) + (z+1)n\,z^{n-1}\} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} [n\,z^{n-1} + n(z+1).(n-1)\,z^{n-2} + n\,z^{n-1}(1)]$$

$$= \frac{1}{2}[n + 2n(n-1) + n]$$

$$= \frac{1}{2}[n + 2n^2 - 2n + n]$$

$$= n^2$$

$$\therefore\ f(n) = \operatorname{Re}s(z=1)$$

$$= n^2$$

19. Solve $y(n+2)+4\,y(n+1)+4\,y(n) = n$ given that y(0) = 0, y(1) = 1 by using Z - transform.

Sol. Given $y(n+2)+4\,y(n+1)+4\,y(n) = n$

Taking Z – transform on both sides, we get

$$Z[y(n+2)] + 4Z[y(n+1)] + 4Z[y(n)] = Z(n)$$

$$\{z^2\bar{y}(z) - z^2y(0) - z\,y(1)\} + 4\{z\,\bar{y}(z) - z\,y(0)\} + 4\,\bar{y}(z) = \frac{z}{(z-1)^2}$$

$$\{z^2\bar{y}(z) - 0 - z(1)\} + 4\{z\,\bar{y}(z) - 0\} + 4\,\bar{y}(z) = \frac{z}{(z-1)^2}$$

$$(z^2+4z+4)\bar{y}(z) = \frac{z}{(z-1)^2} + z$$

$$(z+2)^2\,\bar{y}(z)=\frac{z+z(z-1)^2}{(z-1)^2}$$

$$\bar{y}(z)=\frac{z+z(z-1)^2}{(z-1)^2(z+2)^2}$$

$$\frac{\bar{y}(z)}{z}=\frac{1+(z-1)^2}{(z-1)^2(z+2)^2}=\frac{z^2-2z+2}{(z-1)^2(z+2)^2}$$

$$\frac{z^2-2z+2}{(z-1)^2(z+2)^2}=\frac{A}{z-1}+\frac{B}{(z-1)^2}+\frac{C}{z+2}+\frac{D}{(z+2)^2}$$

$$z^2-2z+2=A(z-1)(z+2)^2+B(z+2)^2+C(z+2)(z-1)^2+D(z-1)^2$$

$$Put\ \ z=1,\ we\ get\qquad 1-2+2=0+B(9)+0+0$$

$$\Rightarrow\ B=\frac{1}{9}$$

$$Put\ \ z=-2,\ we\ get\qquad 4+4+2=0+0+0+D(9)$$

$$\Rightarrow\ D=\frac{10}{9}$$

$$Coeff.\ of\ \ z^3,\qquad 0=A+C\ \ ------\ (1)$$

$$Coeff.\ of\ \ z^2,\qquad 1=3A+B+0+D$$

$$1=3A+\frac{1}{9}+\frac{10}{9}$$

$$1-\frac{11}{9}=3A\ \ \Rightarrow\ A=-\frac{2}{27}$$

$$(1)\Rightarrow\quad 0=-\frac{2}{27}+C$$

$$\Rightarrow\ C=\frac{2}{27}$$

$A(z-1)(z+2)^2$
$A(z-1)(z^2+4z+4)$
$A(z^3+4z^2-z^2+4z-4z-4)$
$A(z^3+3z^2-4)$

$C(z+2)(z-1)^2$
$C(z+2)(z^2-2z+1)$
$C(z^3-2z^2+2z^2+z-4z+2)$
$C(z^3-3z+2)$

$$\frac{\bar{y}(z)}{z}=\frac{-2/27}{z-1}+\frac{1/9}{(z-1)^2}+\frac{2/27}{z+2}+\frac{10/9}{(z+2)^2}$$

$$\bar{y}(z)=-\frac{2}{27}\frac{z}{z-1}+\frac{1}{9}\frac{z}{(z-1)^2}+\frac{2}{27}\frac{z}{z+2}+\frac{10}{9}\frac{z}{(z+2)^2}$$

$$\therefore\ Z^{-1}\{\bar{y}(z)\}=-\frac{2}{27}Z^{-1}\left[\frac{z}{z-1}\right]+\frac{1}{9}Z^{-1}\left[\frac{z}{(z-1)^2}\right]+\frac{2}{27}Z^{-1}\left[\frac{z}{z+2}\right]-\frac{5}{9}Z^{-1}\left[\frac{-2z}{(z+2)^2}\right]$$

$$y(n)=-\frac{2}{27}(1)+\frac{1}{9}(n)+\frac{2}{27}(-2)^n-\frac{5}{9}n.(-2)^n$$

$$\text{(i.e.)}\ \ y(n)=-\frac{2}{27}+\frac{n}{9}+\frac{2}{27}(-2)^n-\frac{5}{9}n.(-2)^n$$

20. Solve $u_{n+2}+6u_{n+1}+9u_n=2^n$ with $u_0=u_1=0$ using Z - transform.

Sol. Given $u_{n+2}+6u_{n+1}+9u_n=2^n$

Taking Z – transform on both sides, we get

$$Z[u_{n+2}]+6Z[u_{n+1}]+9Z[u_n]=Z(2^n)$$

$$\{z^2\bar{u}(z)-z^2u(0)-zu(1)\}+6\{z\bar{u}(z)-zu(0)\}+9\bar{u}(z)=\frac{z}{z-2}$$

$$\{z^2\bar{u}(z)-0-0\}+6\{z\bar{u}(z)-0\}+9\bar{u}(z)=\frac{z}{z-2}$$

$$(z^2+6z+9)\bar{u}(z)=\frac{z}{z-2}$$

$$(z+3)^2\bar{u}(z)=\frac{z}{z-2}$$

$$\bar{u}(z)=\frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{u}(z)}{z}=\frac{1}{(z-2)(z+3)^2}$$

$$\frac{1}{(z-2)(z+3)^2}=\frac{A}{z-2}+\frac{B}{z+3}+\frac{C}{(z+3)^2}$$

$$1=A(z+3)^2+B(z-2)(z+3)+C(z-2)$$

$$Put\ \ z=2,\ we\ get \qquad 1=A(5)^2+0+0$$

$$\Rightarrow\ A=\frac{1}{25}$$

$$Put\ \ z=-3,\ we\ get \qquad 1=0+0+C(-5)$$

$$\Rightarrow\ C=-\frac{1}{5}$$

$$Coeff.\ of\ z^2, \qquad 0=A+B$$

$$0=\frac{1}{25}+B$$

$$\Rightarrow\ B=-\frac{1}{25}$$

$$\frac{\bar{u}(z)}{z}=\frac{1/25}{z-2}-\frac{1/25}{z+3}-\frac{1/5}{(z+3)^2}$$

$$\bar{u}(z)=\frac{1}{25}\frac{z}{z-2}-\frac{1}{25}\frac{z}{z+3}-\frac{1}{5}\frac{z}{(z+3)^2}$$

$$\therefore\ u_n=Z^{-1}\{\bar{u}(z)\}=\frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right]-\frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right]+\frac{1}{15}Z^{-1}\left[\frac{-3z}{(z+3)^2}\right]$$

$$(i.e.)\ u_n=\frac{1}{25}.2^n-\frac{1}{25}(-3)^n+\frac{1}{15}.n(-3)^n$$

21. Solve $u_{n+2}-2u_{n+1}+u_n=2^n$ with $u_0=2,\ u_1=1$ using Z - transform.

Sol. Given $u_{n+2}-2u_{n+1}+u_n=2^n$

Taking Z – transform on both sides, we get

$$Z[u_{n+2}]-2Z[u_{n+1}]+Z[u_n]=Z(2^n)$$

$$\{z^2\bar{u}(z)-z^2u(0)-zu(1)\}-2\{z\bar{u}(z)-zu(0)\}+\bar{u}(z)=\frac{z}{z-2}$$

$$\{z^2\bar{u}(z)-2z^2-z\}-2\{z\bar{u}(z)-2z\}+\bar{u}(z)=\frac{z}{z-2}$$

$$(z^2-2z+1)\bar{u}(z)=\frac{z}{z-2}+2z^2-3z$$

$$(z-1)^2\bar{u}(z)=\frac{z+z(2z-3)(z-2)}{z-2}$$

$$\bar{u}(z)=\frac{z[1+2z^2-7z+6]}{(z-2)(z-1)^2}$$

$$\frac{\bar{u}(z)}{z}=\frac{2z^2-7z+7}{(z-2)(z-1)^2}$$

$$\frac{2z^2-7z+7}{(z-2)(z-1)^2}=\frac{A}{z-2}+\frac{B}{z-1}+\frac{C}{(z-1)^2}$$

$$2z^2-7z+7=A(z-1)^2+B(z-2)(z-1)+C(z-2)$$

$$Put\ \ z=2,\ we\ \ get \qquad 8-14+7=A(1)^2+0+0$$

$$\Rightarrow\ A=1$$

$$Put\ \ z=1,\ we\ \ get \qquad 2-7+7=0+0+C(-1)$$

$$\Rightarrow\ C=-2$$

$$Coeff.\ of\ \ z^2, \qquad 2=A+B$$

$$2=1+B$$

$$\Rightarrow\ B=1$$

$$\frac{\bar{u}(z)}{z}=\frac{1}{z-2}+\frac{1}{z-1}-\frac{2}{(z-1)^2}$$

$$\bar{u}(z)=\frac{z}{z-2}+\frac{z}{z-1}-\frac{2z}{(z-1)^2}$$

$$\therefore\ u_n=Z^{-1}\{\bar{u}(z)\}=Z^{-1}\left[\frac{z}{z-2}\right]+Z^{-1}\left[\frac{z}{z-1}\right]-2Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$(i.e.)\ u_n=2^n+1-2n$$

22. Solve $y_{n+2}+4y_{n+1}+3y_n=3^n$ **with** $y_0=0,\ y_1=1$ **using Z - transform.**

Sol. Given $y_{n+2}+4y_{n+1}+3y_n=3^n$

Taking Z – transform on both sides, we get

$$Z[y_{n+2}]+4Z[y_{n+1}]+3Z[y_n]=Z(3^n)$$

$$\{z^2\bar{y}(z)-z^2y(0)-z\,y(1)\}+4\{z\,\bar{y}(z)-z\,y(0)\}+3\,\bar{y}(z)=\frac{z}{z-3}$$

$$\{z^2\bar{y}(z)-0-z\}+4\{z\,\bar{y}(z)-0\}+3\,\bar{y}(z)=\frac{z}{z-3}$$

$$(z^2+4z+3)\bar{y}(z)=\frac{z}{z-3}+z$$

$$(z+1)(z+3)\bar{y}(z)=\frac{z+z(z-3)}{z-3}$$

$$\bar{y}(z)=\frac{z[1+z-3]}{(z+1)(z+3)(z-3)}$$

$$\bar{y}(z)=\frac{z^2-2z}{(z+1)(z+3)(z-3)}$$

$$\frac{z^2-2z}{(z+1)(z+3)(z-3)}=\frac{A}{z+1}+\frac{B}{z+3}+\frac{C}{z-3}$$

$$z^2-2z=A(z+3)(z-3)+B(z+1)(z-3)+C(z+1)(z+3)$$

$$Put\ \ z=3,\ we\ \ get \qquad 9-6=0+0+C(4)(6)$$

$$\Rightarrow\ C=\frac{3}{24}=\frac{1}{8}$$

$$Put\ \ z=-1,\ we\ \ get \qquad 1+2=A(2)(-4)+0+0$$

$$\Rightarrow\ A=-\frac{3}{8}$$

$$Coeff.\,of\ z^2, \quad 1 = A + B + C$$

$$1 = -\frac{3}{8} + B + \frac{1}{8}$$

$$\Rightarrow\ B = 1 + \frac{3}{8} - \frac{1}{8}$$

$$\Rightarrow\ B = \frac{8+3-1}{8} = \frac{10}{8} = \frac{5}{4}$$

$$\bar{y}(z) = \frac{-3/8}{z+1} + \frac{5/4}{z+3} + \frac{1/8}{z-3}$$

$$\therefore\ y_n = Z^{-1}\{\bar{y}(z)\} = -\frac{3}{8}Z^{-1}\left[\frac{1}{z+1}\right] + \frac{5}{4}Z^{-1}\left[\frac{1}{z+3}\right] + \frac{1}{8}Z^{-1}\left[\frac{1}{z-3}\right]$$

$$y_n = -\frac{3}{8}(-1)^{n-1} + \frac{5}{4}(-3)^{n-1} + \frac{1}{8}(3)^{n-1}$$

$$y_n = -\frac{3}{8}\frac{(-1)^n}{(-1)} + \frac{5}{4}\frac{(-3)^n}{(-3)} + \frac{1}{8}\frac{(3)^n}{3}$$

$$(i.e.)\ y_n = \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{3^n}{24}$$

23.Using Z-transform solve $y(n) + 3\,y(n-1) - 4\,y(n-2) = 0,\ n \geq 2$ **given that** $y(0) = 3,\ y(1) = -2$

Sol. Changing n into n + 2 in the given equation, it becomes

$$y(n+2) + 3\,y(n+1) - 4\,y(n) = 0,\ n \geq 0$$

Taking Z – transform on both sides, we get

$$Z[y(n+2)] + 3Z[y(n+1)] - 4Z[y(n)] = Z(0)$$

$$\{z^2\bar{y}(z) - z^2 y(0) - z\,y(1)\} + 3\{z\,\bar{y}(z) - z\,y(0)\} - 4\,\bar{y}(z) = 0$$

$$\{z^2\bar{y}(z) - 3z^2 + 2z\} + 3\{z\,\bar{y}(z) - 3z\} - 4\,\bar{y}(z) = 0$$

$$(z^2 + 3z - 4)\bar{y}(z) = 3z^2 + 7z$$

$$(z+4)(z-1)\bar{y}(z) = z(3z+7)$$

$$\frac{\bar{y}(z)}{z} = \frac{3z+7}{(z-1)(z+4)}$$

$$\frac{3z+7}{(z-1)(z+4)} = \frac{A}{z-1} + \frac{B}{z+4}$$

$$3z + 7 = A(z+4) + B(z-1)$$

$$Put\ z = 1,\ we\ get \quad 3 + 7 = A(5) + 0$$

$$\Rightarrow\ A = \frac{10}{5} = 2$$

$$Put\ z = -4,\ we\ get \quad -12 + 7 = 0 + B(-5)$$

$$-5 = -5B$$

$$\Rightarrow\ B = 1$$

$$\frac{\bar{y}(z)}{z} = \frac{2}{z-1} + \frac{1}{z+4}$$

$$\bar{y}(z) = \frac{2z}{z-1} + \frac{z}{z+4}$$

$$\therefore\ y_n = Z^{-1}\{\bar{y}(z)\} = 2Z^{-1}\left[\frac{z}{z-1}\right] + Z^{-1}\left[\frac{z}{z+4}\right]$$

$$(i.e.)\ y_n = 2 + (-4)^n$$

24. Using Z-transform method solve $y_{n+2}+y_n=2$ given that $y_0=y_1=0$

Sol. Given $y_{n+2}+y_n=2$

Taking Z – transform on both sides, we get

$$Z[y_{n+2}]+Z[y_n]=Z(2)$$

$$\{z^2\bar{y}(z)-z^2y(0)-z\,y(1)\}+\bar{y}(z)=\frac{2z}{z-1}$$

$$\{z^2\bar{y}(z)-0-0\}+\bar{y}(z)=\frac{2z}{z-1}$$

$$(z^2+1)\bar{y}(z)=\frac{2z}{z-1}$$

$$\bar{y}(z)=\frac{2z}{(z-1)(z^2+1)}$$

$$\frac{\bar{y}(z)}{z}=\frac{2}{(z-1)(z^2+1)}$$

$$\frac{2}{(z-1)(z^2+1)}=\frac{A}{z-1}+\frac{Bz+C}{z^2+1}$$

$$2=A(z^2+1)+(Bz+C)(z-1)$$

$$Put\ \ z=1,\ we\ get\qquad 2=A(2)+0$$

$$\Rightarrow\ A=1$$

$$Coeff.\,of\ z^2,\qquad 0=A+B$$

$$0=1+B$$

$$B=-1$$

$$Coeff.\,of\ z,\qquad 0=-B+C$$

$$0=1+C$$

$$\Rightarrow C=-1$$

(Bz + C)(z – 1)
Bz^2 – Bz +Cz – C

$$\frac{\bar{y}(z)}{z}=\frac{1}{z-1}+\frac{-z-1}{z^2+1}$$

$$\bar{y}(z)=\frac{z}{z-1}-\frac{z^2}{z^2+1}-\frac{z}{z^2+1}$$

$$\therefore\ y_n=Z^{-1}\{\bar{y}(z)\}=Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z^2}{z^2+1}\right]-Z^{-1}\left[\frac{z}{z^2+1}\right]$$

$$(i.e.)\ \ y_n=1-\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)$$

<u>Problems for practice</u>

1. **Solve $y(n+2)-5\,y(n+1)+6\,y(n)=36$ given that y(0) = y(1) = 0 by using Z - transform.**
2. **Using Z-transform method solve $y_{k+2}+2y_{k+1}+y_k=k$ given that $y_0=y_1=0$**
3. **Solve $y(k+2)-4\,y(k+1)+4\,y(k)=0$ given that y(0) = 1, y(1) = 0 by using Z - transform.**
4. **Solve $y(n+3)-3\,y(n+1)+2\,y(n)=0$ given that y(0) = 4, y(1) = 0, y(2) = 8.**

<u>Answer</u>

1. $y(n)=18-36(2)^n+18(3)^n$
2. $y_k=-\frac{1}{4}+\frac{k}{4}+\frac{1}{4}(-1)^k-\frac{1}{4}k\,(-1)^k$
3. $y(k)=2^k-k.2^k$
4. $y(n)=\frac{8}{3}+\frac{4}{3}(-2)^n$

25. Using convolution theorem, find the inverse Z – transform of $\frac{1}{(z-1)(z-2)}$

Sol. $$Z^{-1}\left[\frac{1}{(z-1)(z-2)}\right]=Z^{-1}\left[\frac{1}{z-1}\right]*Z^{-1}\left[\frac{1}{z-2}\right]$$
$$=1*2^{n-1}$$
$$=1*\frac{2^n}{2}=\frac{1}{2}(2^n*1)$$
$$=\frac{1}{2}\sum_{r=0}^{n}2^r.(1)^{n-r}$$
$$=\frac{1}{2}[1+2+2^2+2^3+..........+2^n]$$
$$=\frac{1}{2}\frac{2^{n+1}-1}{2-1}$$
$$=\frac{2^{n+1}-1}{2}$$

26. Using convolution theorem, find the inverse Z – transform of $\frac{z^2}{(z+a)^2}$

Sol. $$Z^{-1}\left[\frac{z^2}{(z+a)^2}\right]=Z^{-1}\left[\frac{z}{z+a}.\frac{z}{z+a}\right]$$
$$=Z^{-1}\left[\frac{z}{z+a}\right]*Z^{-1}\left[\frac{z}{z+a}\right]$$
$$=(-a)^n*(-a)^n$$
$$=\sum_{r=0}^{n}(-a)^r(-a)^{n-r}$$
$$=\sum_{r=0}^{n}(-a)^n$$
$$=(n+1)(-a)^n$$

27. Using convolution theorem, find the inverse Z – transform of $\frac{z^2}{(z+a)(z+b)}$

Sol. $$Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=Z^{-1}\left[\frac{z}{z+a}.\frac{z}{z+b}\right]$$
$$=Z^{-1}\left[\frac{z}{z+a}\right]*Z^{-1}\left[\frac{z}{z+b}\right]$$
$$=(-a)^n*(-b)^n$$
$$=\sum_{r=0}^{n}(-a)^r(-b)^{n-r}$$
$$=(-b)^n\sum_{r=0}^{n}(-a)^r(-b)^{-r}$$
$$=(-b)^n\sum_{r=0}^{n}\left(\frac{-a}{-b}\right)^r$$

$$= (-b)^n \sum_{r=0}^{n} \left(\frac{a}{b}\right)^r$$

$$= (-b)^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \ldots\ldots\ldots + \left(\frac{a}{b}\right)^n\right]$$

$$= (-b)^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \left(\frac{a}{b}\right)}\right]$$

$$= (-1)^n b^n \left[\frac{(b^{n+1} - a^{n+1})/b^{n+1}}{(b-a)/b}\right]$$

$$= (-1)^n \left[\frac{b^{n+1} - a^{n+1}}{b-a}\right]$$

28. Using convolution theorem, find the inverse Z – transform of $\frac{12z^2}{(3z-1)(4z+1)}$

Sol. $Z^{-1}\left[\frac{12z^2}{(3z-1)(4z+1)}\right] = Z^{-1}\left[\frac{12z^2}{3\left(z-\frac{1}{3}\right)4\left(z+\frac{1}{4}\right)}\right] = Z^{-1}\left[\frac{z^2}{\left(z-\frac{1}{3}\right)\left(z+\frac{1}{4}\right)}\right]$

$$= Z^{-1}\left[\frac{z}{z-1/3}\right] * Z^{-1}\left[\frac{z}{z+1/4}\right]$$

$$= (1/3)^n * (-1/4)^n$$

$$= (-1/4)^n * (1/3)^n$$

$$= \sum_{r=0}^{n} (-1/4)^r (1/3)^{n-r}$$

$$= \left(\frac{1}{3}\right)^n \sum_{r=0}^{n} \left(\frac{-1}{4}\right)^r (3)^r$$

$$= \left(\frac{1}{3}\right)^n \sum_{r=0}^{n} \left(\frac{-3}{4}\right)^r$$

$$= \left(\frac{1}{3}\right)^n \left[1 + \left(\frac{-3}{4}\right) + \left(\frac{-3}{4}\right)^2 + \ldots\ldots\ldots + \left(\frac{-3}{4}\right)^n\right]$$

$$= \left(\frac{1}{3}\right)^n \left[\frac{1 - \left(\frac{-3}{4}\right)^{n+1}}{1 - \left(\frac{-3}{4}\right)}\right]$$

$$= \left(\frac{1}{3}\right)^n \frac{4}{7}\left[1 - \left(\frac{-3}{4}\right)^n \left(\frac{-3}{4}\right)\right]$$

$$= \left(\frac{1}{3}\right)^n \left[\frac{4}{7} + \frac{3}{7}\left(\frac{-3}{4}\right)^n\right]$$

29. Using convolution theorem, find the inverse Z – transform of $\dfrac{z^2}{(z-4)(z-3)}$

Sol.
$$Z^{-1}\left[\frac{z^2}{(z-4)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-4}.\frac{z}{z-3}\right]$$
$$= Z^{-1}\left[\frac{z}{z-4}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$
$$= (4)^n * (3)^n$$
$$= \sum_{r=0}^{n} (4)^r\,(3)^{n-r}$$
$$= 3^n \sum_{r=0}^{n} (4)^r (3)^{-r}$$
$$= 3^n \sum_{r=0}^{n} \left(\frac{4}{3}\right)^r$$
$$= 3^n \left[1+\left(\frac{4}{3}\right)+\left(\frac{4}{3}\right)^2+.........+\left(\frac{4}{3}\right)^n\right]$$
$$= 3^n \left[\frac{(4/3)^{n+1}-1}{(4/3)-1}\right]$$
$$= 3^n \left[\frac{(4^{n+1}-3^{n+1})/3^{n+1}}{1/3}\right]$$
$$= 4^{n+1} - 3^{n+1}$$

30. Using convolution theorem, find the inverse Z – transform of $\left(\dfrac{z}{z-4}\right)^3$

Sol.
$$Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right] = Z^{-1}\left[\left(\frac{z}{z-4}\right)^2\right] * Z^{-1}\left[\left(\frac{z}{z-4}\right)\right] \text{ ------ (1)}$$
$$Z^{-1}\left[\left(\frac{z}{z-4}\right)^2\right] = Z^{-1}\left[\left(\frac{z}{z-4}\right)\right] * Z^{-1}\left[\left(\frac{z}{z-4}\right)\right]$$
$$= 4^n * 4^n$$
$$= \sum_{r=0}^{n} (4)^r\,(4)^{n-r} = \sum_{r=0}^{n} (4)^n$$
$$= (n+1)(4)^n$$

Equation (1) becomes
$$Z^{-1}\left[\left(\frac{z}{z-4}\right)^3\right] = (n+1)\,4^n * 4^n$$
$$= \sum_{r=0}^{n} (r+1)(4)^r\,(4)^{n-r}$$
$$= \sum_{r=0}^{n} (r+1)(4)^n$$
$$= 4^n[1+2+3+.........+(n+1)]$$
$$= 4^n\,\frac{(n+1)(n+2)}{2}$$

31. Form the difference equation whose solution is $y_n = (A+Bn)2^n$

Sol. Given $y_n = (A+Bn)2^n = A2^n + Bn2^n$ ------------- (1)

$$y_{n+1} = [A+B(n+1)]2^{n+1} = 2[A+B(n+1)]2^n$$
$$= 2A2^n + 2B(n+1)2^n \quad ------- (2)$$

$$y_{n+2} = [A+B(n+2)]2^{n+2} = 4[A+B(n+2)]2^n$$
$$= 4A2^n + 4B(n+2)2^n \quad ------- (3)$$

Eliminating A and B from equations (1), (2) and (3), we have

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$y_n[8(n+2)-8(n+1)] - y_{n+1}[4(n+2)-4n] + y_{n+2}[2(n+1)-2n] = 0$$
$$y_n(16-8) - y_{n+1}(8) + y_{n+2}(2) = 0$$
$$8y_n - 8y_{n+1} + 2y_{n+2} = 0$$
$$(i.e.)\ y_{n+2} - 4y_{n+1} + 4y_n = 0$$

32. Derive the difference equation from $y_n = (A+Bn)(-3)^n$

Sol. Given $y_n = (A+Bn)(-3)^n = A(-3)^n + Bn(-3)^n$ ------------- (1)

$$y_{n+1} = [A+B(n+1)](-3)^{n+1} = -3[A+B(n+1)](-3)^n$$
$$= -3A(-3)^n - 3B(n+1)(-3)^n \quad ------- (2)$$

$$y_{n+2} = [A+B(n+2)](-3)^{n+2} = 9[A+B(n+2)](-3)^n$$
$$= 9A(-3)^n + 9B(n+2)(-3)^n \quad ------- (3)$$

Eliminating A and B from equations (1), (2) and (3), we have

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & -3 & -3(n+1) \\ y_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0$$

$$y_n[-27(n+2)+27(n+1)] - y_{n+1}[9(n+2)-9n] + y_{n+2}[-3(n+1)+3n] = 0$$
$$y_n(-54+27) - y_{n+1}(18) + y_{n+2}(-3) = 0$$
$$-27y_n - 18y_{n+1} - 3y_{n+2} = 0$$
$$(i.e.)\ y_{n+2} + 6y_{n+1} + 9y_n = 0$$

33. Find $Z^{-1}\left[\dfrac{z^2-3z}{z^3-3z^2+4}\right]$ **using partial fraction method.**

Sol. $Let\ \bar{f}(z) = \dfrac{z^2-3z}{z^3-3z^2+4} = \dfrac{z^2-3z}{(z+1)(z-2)^2}$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & 0 & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

(z + 1) is a factor.
The other factors are
$z^2 - 4z + 4$
$(z - 2)^2$

$$\frac{\bar{f}(z)}{z} = \frac{z-3}{(z+1)(z-2)^2} = \frac{A}{z+1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$z-3 = A(z-2)^2 + B(z-2)(z+1) + C(z+1)$$

$$Put\ z=-1,\ we\ get \quad -4 = A(-3)^2 + 0 + 0$$

$$\Rightarrow A = -\frac{4}{9}$$

$$Put\ z=2,\ we\ get \quad -1 = 0 + 0 + C(3)$$

$$\Rightarrow C = -\frac{1}{3}$$

$$Coeff.\,of\;\; z^2, \qquad 0 = A + B$$

$$0 = -\frac{4}{9} + B \quad \Rightarrow B = \frac{4}{9}$$

$$\frac{\bar{f}(z)}{z} = \frac{-4/9}{z+1} + \frac{4/9}{z-2} + \frac{-1/3}{(z-2)^2}$$

$$\bar{f}(z) = -\frac{4}{9}\frac{z}{z+1} + \frac{4}{9}\frac{z}{z-2} - \frac{1}{3}\frac{z}{(z-2)^2}$$

$$\therefore\; Z^{-1}\{\bar{f}(z)\} = -\frac{4}{9}Z^{-1}\left[\frac{z}{z+1}\right] + \frac{4}{9}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{6}Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= -\frac{4}{9}(-1)^n + \frac{4}{9}(2)^n - \frac{1}{6}(n.2^n)$$

34. Using convolution theorem, find the inverse Z – transform of $\dfrac{z^3}{(z-2)^2(z-3)}$

Sol. $$Z^{-1}\left[\frac{z^3}{(z-2)^2(z-3)}\right] = Z^{-1}\left[\frac{z^2}{(z-2)^2}.\frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z^2}{(z-2)^2}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (n+1)(2)^n * (3)^n$$

$$= \sum_{r=0}^{n}(r+1)(2)^r\,(3)^{n-r}$$

$$= 3^n \sum_{r=0}^{n}(r+1)(2)^r(3)^{-r}$$

$$= 3^n \sum_{r=0}^{n}(r+1)\left(\frac{2}{3}\right)^r$$

$$Let\;\; S = 1 + 2x + 3x^2 + 4x^3 + ...+(n+1)x^n$$

$$xS = x + 2x^2 + 3x^3 + ...+ n\,x^n + (n+1)x^{n+1}$$

$$S - xS = 1 + x + x^2 + x^3 + ...+ x^n - (n+1)x^{n+1}$$

$$(1-x)S = \frac{1-x^{n+1}}{1-x} - (n+1)x^{n+1}$$

$$S = \frac{1-x^{n+1}}{(1-x)^2} - \frac{(n+1)x^{n+1}}{1-x}$$

$$= 3^n\left[1 + 2\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^3 + ...+(n+1)\left(\frac{2}{3}\right)^n\right]$$

$$= 3^n\left[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{\left(1-\frac{2}{3}\right)^2} - \frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}\right]$$

$$= 3^n\left[\frac{1-\left(\frac{2}{3}\right)^n\left(\frac{2}{3}\right)}{\frac{1}{9}} - \frac{(n+1)\left(\frac{2}{3}\right)^n\left(\frac{2}{3}\right)}{\frac{1}{3}}\right]$$

$$= 3^n\left[9\left\{1-\left(\frac{2}{3}\right)^n\left(\frac{2}{3}\right)\right\} - 3\left\{(n+1)\left(\frac{2}{3}\right)^n\left(\frac{2}{3}\right)\right\}\right]$$

$$= 3^n\left[9 - 6\left(\frac{2}{3}\right)^n - 2\,(n+1)\left(\frac{2}{3}\right)^n\right]$$

$$= 3^n\left[9 - \left(\frac{2}{3}\right)^n(6 + 2n + 2)\right]$$

$$= 3^n\left[9 - \left(\frac{2}{3}\right)^n(2n+8)\right]$$

$$= 9.3^n - 2^n(2n+8)$$

35. Using convolution theorem, find the inverse Z – transform of $\dfrac{z^2}{(z-1)^2(z-2)}$

Sol. $Z^{-1}\left[\dfrac{z^2}{(z-1)^2(z-2)}\right] = Z^{-1}\left[\dfrac{z}{(z-1)^2}.\dfrac{z}{z-2}\right]$

$$= Z^{-1}\left[\frac{z}{(z-1)^2}\right] * Z^{-1}\left[\frac{z}{z-2}\right]$$

$$= n(1)^n * (2)^n = n * 2^n$$

$$= \sum_{r=0}^{n} r\,(2)^{n-r}$$

$$= 2^n \sum_{r=0}^{n} r(2)^{-r}$$

$$= 2^n \sum_{r=0}^{n} r\left(\frac{1}{2}\right)^r$$

$$= 2^n\left[0+1\left(\frac{1}{2}\right)+2\left(\frac{1}{2}\right)^2+3\left(\frac{1}{2}\right)^3+.........+n\left(\frac{1}{2}\right)^n\right]$$

$$= 2^n\left(\frac{1}{2}\right)\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+.........+n\left(\frac{1}{2}\right)^{n-1}\right]$$

$$= \frac{2^n}{2}\left[\frac{1-\left(\frac{1}{2}\right)^n}{\left(1-\frac{1}{2}\right)^2} - \frac{n\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right]$$

$$\begin{aligned}
&Let\ \ S = 1+2x+3x^2+4x^3+...........+n\,x^{n-1}\\
&xS = x+2x^2+3x^3+...........+(n-1)x^{n-1}+n\,x^n\\
&S-xS = 1+x+x^2+x^3+...........+x^{n-1}-n\,x^n\\
&(1-x)S = \frac{1-x^n}{1-x}-n\,x^n\\
&S = \frac{1-x^n}{(1-x)^2}-\frac{n\,x^n}{1-x}
\end{aligned}$$

$$= \frac{2^n}{2}\left[\frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{4}} - \frac{n\left(\frac{1}{2}\right)^n}{\frac{1}{2}}\right]$$

$$= \frac{2^n}{2}\left[4\left\{1-\left(\frac{1}{2}\right)^n\right\}-2\left\{n\left(\frac{1}{2}\right)^n\right\}\right]$$

$$= \frac{2^n}{2}\left[4-4\left(\frac{1}{2}\right)^n-2n\left(\frac{1}{2}\right)^n\right]$$

$$= 2.2^n - 2 - n$$

UNIT – I

Probability and Random Variables.

Definition : Let S be the sample space and A be an event associated with a random experiment. Let n(S) and n(A) be the no. of elements of S and A. Then the probability of event A occurring denoted as p(A) is defined as

$$p(A) = \frac{n(A)}{n(S)} = \frac{\text{No. of favourable cases to A}}{\text{Total no. of sample space in S}}.$$

---

1) What is the probability of getting an even no. in the die tossing experiment?

Sol.

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$n(S) = 6.$$

$$A = \{2, 4, 6\}, \quad n(A) = 3.$$

$$\therefore p(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2} = 0.5$$

Axioms of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A denoted by p(A) is defined as a real no. satisfying the following axioms.

i) $0 \leq P(A) \leq 1$ (or) $P(A) \geq 0$

ii) $P(S) = 1$ where S is a sure event (sample space)

Results

1) $P(\phi) = 0$ [i.e. probability of an impossible event is 0]

2) $P(\bar{A}) = 1 - P(A)$

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [Addition theorem]

4) If two events A and B are independent, then

$$P(A \cap B) = P(A) \cdot P(B)$$

5) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$; $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Conditional probability.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B) \cdot P(B)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A) \cdot P(A)$$

1) If A and B are the events such that $P(A \cup B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$, $P(\bar{A}) = \frac{2}{3}$. Determine $P(\bar{A}/B)$

Sol. $$P(\bar{A}/B) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)} \quad \text{——— } ①$$

Given $P(\bar{A}) = \frac{2}{3} \Rightarrow P(A) = 1 - \frac{2}{3} = \frac{1}{3}$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{3}{4} = \frac{1}{3} + P(B) - \frac{1}{4}$$

$$P(B) = \frac{3}{4} + \frac{1}{4} - \frac{1}{3} = \frac{2}{3}.$$

$$① \Rightarrow p(\bar{A}/B) = \frac{2/3 - 1/4}{2/3} = \frac{\frac{8-3}{12}}{2/3}$$

$$= \frac{5}{12} \cdot \frac{3}{2} = \frac{5}{8}$$

Theorem of Total probability

If $E_1, E_2, \ldots E_n$ be a set of exhaustive and mutually exclusive events and A is another event associated with E_i then $p(A) = \sum_{i=1}^{n} p(E_i)\, p(A/E_i)$

Baye's theorem (or) Theorem of Probability of Causes

If $E_1, E_2, \ldots E_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with E_i then $p(E_i/A) = \dfrac{p(E_i)\, p(A/E_i)}{\sum_{i=1}^{n} p(E_i)\, p(A/E_i)}$

1) A company has two plants. plant 1 manufactures 25% of the items. plant 2 manufactures 75% of the items. 3% and 5% of the items manufactured by plant 1 and plant 2 respectively are found to be defective. An item is selected at random and is known to be defective. What is the chance that it was generated by plant 2.

Sol. plant 1. → E_1 Plant 2 → E_2

$$P(E_1) = \frac{25}{100} = 0.25 \qquad P(E_2) = \frac{75}{100} = 0.75$$

Let A be the defective

$$P(A/E_1) = \frac{3}{100} = 0.03$$

$$P(A/E_2) = \frac{5}{100} = 0.05$$

Using Baye's theorem

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{2} P(E_i)P(A/E_i)}$$

$$\therefore P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{0.75(0.05)}{0.25(0.03)+0.75(0.05)}$$

$$= 0.83$$

2) A toy is rejected if the design is faulty or not. The probability that the design is faulty is 0.1 and that the toy is rejected if the design is faulty is 0.95 and otherwise 0.45. If a toy is rejected, what is the probability that it is due to faulty design?

Sol. Let E_1 be the event that the design is faulty.

E_2 be the " " is not faulty.

A denote the event that the toy is rejected.

$p(E_1) = 0.1 \Rightarrow p(E_2) = 1 - 0.1 = 0.9$

$p(A/E_1) = 0.95, \quad p(A/E_2) = 0.45$

Using Baye's theorem,

$$P(E_1/A) = \frac{p(E_1)p(A/E_1)}{p(E_1)p(A/E_1) + p(E_2)p(A/E_2)}$$

$$= \frac{0.1(0.95)}{0.1(0.95) + 0.9(0.45)}$$

$$= 0.19$$

3) A box contains 2000 components of which 15% are defective. A second box contains 5000 components of which 25% are defective. Two other boxes contains 1000 components each with 10% defectives. A box is chosen at random and the event is selected and was found to be defective. Find the chance that it came from second box.

Sol. Probability of selecting a box $\Rightarrow p(E_1) = p(E_2)$

$= p(E_3) = p(E_4) = \frac{1}{4}$

Let A be the defective

$\therefore p(A/E_1) = 0.15$, $p(A/E_2) = 0.25$, $p(A/E_3) = 0.10$,

$p(A/E_4) = 0.10$

By Baye's theorem,

$$p(E_2/A) = \frac{p(E_2)\,p(A/E_2)}{p(E_1)\,p(A/E_1) + p(E_2)\,p(A/E_2) + p(E_3)\,p(A/E_3) + p(E_4)\cdot p(A/E_4)}$$

$$= \frac{(0.25)(0.25)}{(0.25)(0.15) + (0.25)(0.25) + (0.25)(0.10) + 0.25(0.10)}$$

$$= 0.417$$

Random Variables (RV)

A random variable is a fn. that assigns a real no. $X(s)$ to every element $s \in S$, where S is the sample space corresponding to a random experiment.

A real valued fn. defined on S and taking values in $(-\infty, \infty)$ is called a one dimensional random variable. If the values are ordered pairs of real no.'s, the fn. is said to be two dimensional random variable.

Eg: Tossing 2 coins simultaneously, we get HH, HT, TH, TT

Let the no. of heads (or tails) be denoted as $X \longrightarrow$ RV

Its probability distribution is

X :	0	1	2
$p(x)$:	$1/4$	$2/4$	$1/4$

Discrete RV

A random variable which takes a countable no. of values is called a discrete RV.

Continuous RV

A random variable X is said to be continuous if it can take countably infinite values in an interval.

Probability mass function (Discrete)

Let X be a one dimensional discrete random variable which takes the values $x_1, x_2, \ldots$ To each possible outcomes x_i we can associate a no. $p_i = p(x_i) = p(X = x_i)$ called the probability mass fn. of x_i. The fn. $p(x_i)$

satisfies the following conditions.

i) $p(x_i) \geq 0 \quad \forall \; i = 1, 2, \ldots \infty$

ii) $\sum_{i=1}^{\infty} p(x_i) = 1.$

Probability density fn. (Continuous)

The probability fn. of a continuous RV X is called as probability density fn. (pdf) if

i) $f(x) \geq 0 \quad \forall x$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Note:- (Only in continuous case)

$$p(a \leq x \leq b) = p(a < x < b) = \int_{a}^{b} f(x)\,dx$$

Problems.

1) Verify whether the following is a probability distribution.

X :	1	2	3	4	5
p(x) :	$1/10$	$2/10$	$4/10$	$2/10$	$1/10$

Sol.

i) $p(x) \geq 0 \quad \forall x$

ii) $\sum p(x) = \frac{1}{10} + \frac{2}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} = \frac{10}{10} = 1$

Hence the given fn. is a probability distribution.

2) A die is rolled in such a way that each odd no. is twice as likely to occur as each even no. Find $P(G)$ where G is the event that a no. > 3 occurs on a single roll of a die.

Sol. Probability distribution is

X:	1	2	3	4	5	6
$P(X)$:	$2x$	x	$2x$	x	$2x$	x

Since $\sum p(x) = 1$, we have

$$2x + x + 2x + x + 2x + x = 1$$

$$9x = 1$$

$$x = \frac{1}{9}$$

Hence

X:	1	2	3	4	5	6
$P(X)$:	$2/9$	$1/9$	$2/9$	$1/9$	$2/9$	$1/9$

$$P(X>3) = P(X=4) + P(X=5) + P(X=6)$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}$$

3) If X is the discrete RV having the following probability distribution.

X:	0	1	2	3	4	5	6	7
$P(X)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Determine the constant K, $P(X<6)$, $P(X \geq 6)$, $P(0<X<5)$

Sol. Since $\sum p(x) = 1$, we have

$$0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2 + 9K - 1 = 0.$$

Solving, we get $K = 0.1, -1$

$\therefore K = 0.1 = \frac{1}{10}$

X:	0	1	2	3	4	5	6	7
$p(x)$:	0	$1/10$	$2/10$	$2/10$	$3/10$	$1/100$	$2/100$	$17/100$

Now, $p(X<6) = p(X=0) + p(X=1) + \dots + p(X=5)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}$

$= 0.81$

$p(X \geq 6) = p(X=6) + p(X=7)$

$= \frac{2}{100} + \frac{17}{100} = \frac{19}{100} = 0.19$

$p(0<X<5) = p(X=1) + p(X=2) + p(X=3) + p(X=4)$

$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10}$

$= \frac{8}{10} = 0.8$

4) Given $p(x) = \begin{cases} x/15, & x = 1,2,3,4,\dots \\ 0 & \text{otherwise} \end{cases}$

Find i) $p\{X=1 \text{ or } X=2\}$ ii) $p\{\frac{1}{2} < X < \frac{5}{2} / X>1\}$

Sol: i) $p\{X=1 \text{ or } X=2\} = p(X=1) + p(X=2)$

$= \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$

ii) $P\left\{\frac{1}{2} < x < \frac{5}{2} / x > 1\right\} = \frac{P\left[\left(\frac{1}{2} < x < \frac{5}{2}\right) \cap (x > 1)\right]}{P(x > 1)}$

$$= \frac{P(x=2)}{1 - P(x \le 1)} = \frac{P(x=2)}{1 - P(x=1)}$$

$$= \frac{2/15}{1 - \frac{1}{15}} = \frac{2}{14} = \frac{1}{7}$$

5) If the fn is pdf evaluate the value of c

$$f(x) = \begin{cases} 0, & x < 2 \\ c(3+2x), & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

Sol. Since the fn is pdf, we have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{2}^{4} c(3+2x)\,dx = 1$$

$$c\left[3x + \frac{2x^2}{2}\right]_2^4 = 1$$

$$c\left[(12+16) - (6+4)\right] = 1$$

$$c(18) = 1$$

$$c = \frac{1}{18}$$

6) Given

x:	-3	-2	-1	0	1	2	3
$p(x)$:	$1/6$	$1/6$	$1/6$	$1/12$	$1/12$	$1/6$	$\frac{1}{6}$

Find i) $P[|x| \le 1]$ ii) $P[|x| > 2]$ iii) $P[(2x+3) \le 5]$

Sol. i) $p[|x| \leq 1] = p(-1 \leq x \leq 1)$

$= p(x=-1) + p(x=0) + p(x=1)$

$= \frac{1}{6} + \frac{1}{12} + \frac{1}{12}$

$= \frac{1}{3}$.

ii) $p[|x| > 2] = p(x=-3) + p(x=3)$

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

iii) $p[(2x+3) \leq 5] = p[2x \leq 5-3]$

$= p[2x \leq 2]$

$= p[x \leq 1]$

$= p(x=-3) + \cdots\cdots + p(x=1)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12}$

$= \frac{4}{6} = \frac{2}{3}$

---

7) If the RV takes the value 1, 2, 3 and 4 such that $2p(x=1) = 3p(x=2) = p(x=3) = 5p(x=4)$. Find the probability distribution.

Sol.

$2p(x=1) = p(x=3)$

$\Rightarrow p(x=1) = \frac{1}{2} p(x=3)$

$3p(x=2) = p(x=3)$

$\Rightarrow p(x=2) = \frac{1}{3} p(x=3)$

$5p(x=4) = p(x=3)$

$\Rightarrow p(x=4) = \frac{1}{5} p(x=3)$

Since $\Sigma p(x) = 1$, we have

$$p(x=1) + p(x=2) + p(x=3) + p(x=4) = 1$$

$$\frac{1}{2} p(x=3) + \frac{1}{3} p(x=3) + p(x=3) + \frac{1}{5} p(x=3) = 1$$

$$p(x=3)\left[\frac{1}{2} + \frac{1}{3} + 1 + \frac{1}{5}\right] = 1$$

$$p(x=3)\left(\frac{61}{30}\right) = 1$$

$$p(x=3) = \frac{30}{61}$$

$$\therefore p(x=1) = \frac{1}{2}\left(\frac{30}{61}\right) = \frac{15}{61}$$

$$p(x=2) = \frac{1}{3}\left(\frac{30}{61}\right) = \frac{10}{61}$$

$$p(x=4) = \frac{1}{5}\left(\frac{30}{61}\right) = \frac{6}{61}$$

Probability distribution is

x :	1	2	3	4
$p(x)$:	$15/61$	$10/61$	$30/61$	$\frac{6}{61}$

---

8) Given pdf $f(x) = \begin{cases} Ax(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Find the value of A and find k such that $p(x \leq k) = p(x \geq k)$

Sol: Since the fn is pdf, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 Ax(1-x)\,dx = 1$$

$$A\int_0^1 (x - x^2)\,dx = 1$$

$$A\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 = 1$$

$$A\left[\left(\frac{1}{2}-\frac{1}{3}\right)-0\right]=1$$

$$A\left(\frac{1}{6}\right)=1$$

$$A=6.$$

Given $p(X \leq k) = p(X \geq k)$

$$\int_0^K f(x)dx = \int_K^1 f(x)dx$$

$$\int_0^K 6(x-x^2)dx = \int_K^1 6(x-x^2)dx$$

$$\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^K = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_K^1$$

$$\left(\frac{K^2}{2}-\frac{K^3}{3}\right)-0 = \left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{K^2}{2}-\frac{K^3}{3}\right)$$

$$\frac{2K^2}{2}-\frac{2K^3}{3} = \frac{1}{6}$$

$$\frac{6K^2-4K^3}{6} = \frac{1}{6}$$

$$4K^3-6K^2+1=0.$$

Solving, we get

$K=0.5, 1.3, -0.3$

$\therefore K=0.5$

Cumulative Distribution Function (CDF)

If X is an RV discrete or continuous then $F(x) = p(X \leq x)$ is called CDF of X (or) distribution fn of X.

If x is continuous, $F(x) = p(X \leq x) = \int_{-\infty}^{x} f(x)dx$

Properties of CDF

i) $F(-\infty) = 0$

ii) $F(\infty) = 1$

iii) $\frac{d}{dx} F(x) = f(x)$

(ie) $F'(x) = f(x)$

Note :- If $F(x)$ is given then $p(a < x < b) = F(b) - F(a)$.

1) A RV X has the following distribution

X :	-2	-1	0	1	2	3
p(X) :	0.1	k	0.2	2k	0.3	3k

Find k and CDF of X.

Sol. Since $\Sigma p(x) = 1$, we have

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$6k = 1 - 0.6$$

$$= 0.4$$

$$k = \frac{1}{15}$$

X :	-2	-1	0	1	2	3
p(X) :	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{1}{5}$

To find CDF.

X	$F(x) = p(X \le x)$
-2	$F(-2) = p(X \le -2) = \frac{1}{10}$
-1	$F(-1) = p(X \le -1) = \frac{1}{10} + \frac{1}{15} = \frac{1}{6}$
0	$F(0) = p(X \le 0) = \frac{1}{6} + \frac{1}{5} = \frac{11}{30}$
1	$F(1) = p(X \le 1) = \frac{11}{30} + \frac{2}{15} = \frac{1}{2}$
2	$F(2) = p(X \le 2) = \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$
3	$F(3) = p(X \le 3) = \frac{4}{5} + \frac{1}{5} = 1$

2) If pdf of a RV is given by $f(x) = \begin{cases} K(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find K and distribution fn.

Sol. Since the fn is pdf, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 K(1-x^2)\,dx = 1$$

$$K\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$K\left[\left(1 - \frac{1}{3}\right) - 0\right] = 1$$

$$K\left(\frac{2}{3}\right) = 1$$

$$K = \frac{3}{2}.$$

To find CDF.

$$F(x) = p(X \le x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x K(1-x^2)\,dx.$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^x = \frac{3}{2}\left(x - \frac{x^3}{3}\right)$$

Sol: As there is no x terms in the distribution fn given is a discrete RV.

i) Hence the probability distribution is given by

x :	1	4	6	10
$p(x)$:	$1/3$	$\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$	$\frac{5}{6}-\frac{1}{2}=\frac{1}{3}$	$1-\frac{5}{6}=\frac{1}{6}$

ii) $p(2<x<6) = p(x=4) = \frac{1}{6}$

iii) $\text{Mean} = E(x) = \sum x\, p(x)$

$$= \left(1\times\frac{1}{3}\right)+\left(4\times\frac{1}{6}\right)+\left(6\times\frac{1}{3}\right)+\left(10\times\frac{1}{6}\right)$$

$$= \frac{14}{3}.$$

$$E(x^2) = \sum x^2 p(x)$$

$$= \left(1\times\frac{1}{3}\right)+\left(16\times\frac{1}{6}\right)+\left(36\times\frac{1}{3}\right)+\left(100\times\frac{1}{6}\right)$$

$$= \frac{95}{3}.$$

iv) $\text{Variance} = E(x^2) - (E(x))^2$

$$= \frac{95}{3} - \frac{196}{9}$$

$$= \frac{89}{9}.$$

3) If a RV X has a CDF F(x) given by

$$F(x) = \begin{cases} 0, & \text{if } x \leq 1 \\ c(x-1)^4, & 1 \leq x \leq 3 \\ 1, & x > 3. \end{cases}$$

Find f(x), value of c and $P(1 < X < 2)$.

Sol.

$$f(x) = F'(x)$$

$$= \begin{cases} 0, & \text{if } x \leq 1 \\ 4c(x-1)^3, & \text{if } 1 < x \leq 3 \\ 0, & \text{if } x > 3. \end{cases}$$

Since the fn f(x) is pdf, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_1^3 4c(x-1)^3\,dx = 1$$

$$4c\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$c[16-0] = 1 \Rightarrow c = \frac{1}{16}.$$

$$P(1 < X < 2) = F(2) - F(1)$$

$$= \frac{1}{16}(2-1)^4 - 0 = \frac{1}{16}.$$

4) If X has the distribution fn $F(x) = \begin{cases} 0, & x < 1 \\ 1/3, & 1 \leq x < 4 \\ 1/2, & 4 \leq x < 6 \\ 5/6, & 6 \leq x < 10 \\ 1, & x \geq 10 \end{cases}$

Find i) The prob. dist. of X

ii) $P(2 < X < 6)$ iii) Mean iv) Variance.

Expectations or Mean of X.

The averaging process when applying to the RV is called expectations. It is denoted by $E(X)$ (or) μ_1' (or) $\bar{X}$

Formula. (Discrete Case)

$$\text{Mean} = E(x) = \mu_1' = \sum x\, p(x)$$

$$E(x^2) = \mu_2' = \sum x^2 p(x)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$E(x^r) = \mu_r' = \sum x^r p(x)$$

Continuous Case.

$$\text{Mean} = E(x) = \mu_1' = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(x^2) = \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$\vdots$$

$$E(x^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$\mu_1', \mu_2', \ldots \mu_r'$ are called moments about origin (Raw moments)

$\mu_1, \mu_2, --- \mu_r$ are called moments about mean (central moments)

In discrete case, moments about mean is

$$\mu_r = \sum (x - \bar{x})^r p(x)$$

In continuous case, moments about mean is

$$\mu_r = \int_{-\infty}^{\infty} (x - \bar{x})^r f(x)\, dx$$

Properties of Expectations.

i) $E(a) = a$

ii) $E(ax) = a\,E(x)$

iii) $E(ax+b) = a\,E(x) + b$

iv) $E(x \pm y) = E(x) \pm E(y)$

v) $E(xy) = E(x) \cdot E(y)$

Variance. (Second moment about mean, μ_2)

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \mu_2' - (\mu_1')^2$$

S.D. (Standard deviation), $\sigma = \sqrt{Var(x)}$

Properties of Variance.

i) $Var(x) \geq 0$

ii) $Var(b) = 0$

iii) $Var(ax \pm b) = a^2\,Var(x)$

iv) $Var(ax+by) = a^2\,Var(x) + b^2\,Var(y)$

---

1) Given

x :	-3	-2	-1	0	1	2	3
$p(x)$:	0.05	0.1	0.3	0	0.3	0.15	0.1

Find i) $E(x)$ ii) $E(2x \pm 3)$ iii) $E(x^2)$ iv) $Var(x)$, σ

v) $Var(2x \pm 3)$

Sol. i) $E(x) = \sum x\,p(x)$

$$= (-3 \times 0.05) + (-2 \times 0.1) + (-1 \times 0.3) + 0 + (1 \times 0.3) + (2 \times 0.15) + (3 \times 0.1)$$

$$= 0.25$$

ii) $E(2x \pm 3) = 2E(x) \pm 3$

$= 2(0.25) \pm 3$

$= 0.5 \pm 3$

$= 3.5, -2.5$

iii) $E(x^2) = \sum x^2 p(x)$

$= (9 \times 0.05) + (4 \times 0.1) + (1 \times 0.3) + 0 + (1 \times 0.3)$

$+ (4 \times 0.15) + (9 \times 0.1)$

$= 2.95$

iv) $Var(x) = E(x^2) - (E(x))^2$

$= 2.95 - (0.25)^2$

$= 2.8875$

$S.D, \sigma = \sqrt{Var(x)} = \sqrt{2.8875} = 1.699$

v) $Var(2x \pm 3) = 4\,Var(x) + 0$

$= 4(2.8875)$

$= 11.55$

---

2) A continuous RV has a pdf of $f(x) = kx^2 e^{-x}, x \geq 0$

Find k, Mean, Variance.

Sn. Since the fn. is pdf, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} kx^2 e^{-x}\,dx = 1$$

$$k\int_0^{\infty} x^2 e^{-x}\,dx = 1$$

$$k(2!) = 1$$

$$k = \frac{1}{2}$$

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\,dx$$

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x}\,dx$$

$$\Gamma(n+1) = n!$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\infty} x \frac{1}{2} x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} x^3 e^{-x}\,dx$$

$$= \frac{3!}{2} = \frac{6}{2} = 3$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} x^4 e^{-x}\,dx.$$

$$= \frac{4!}{2} = \frac{24}{2} = 12.$$

$$\text{Var}(x) = E(x^2) - (E(x))^2$$

$$= 12 - (3)^2$$

$$= 3.$$

Moment Generating Function (MGF)

MGF of the RV X is

$$M_X(t) = E[e^{tx}] = \sum e^{tx} p(x) \quad \text{— Discrete}$$

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad \text{— Continuous}$$

Discrete case.

$$\text{Mean} = E(x) = \mu_1' = [M_X'(t)]_{t=0}$$

$$E(x^2) = [M_X''(t)]_{t=0}$$

$$E(x^r) = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$$

Continuous case.

$$\text{Mean} = E(x) = \text{Coeff of } t \text{ in } M_X(t)$$

$$E(x^2) = \text{Coeff. of } \frac{t^2}{2!} \text{ in } M_X(t)$$

$$E(x^r) = \text{Coeff. of } \frac{t^r}{r!} \text{ in } M_X(t)$$

i) Find the MGF of $f(x) = \begin{cases} \frac{1}{2} e^{-x/2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Hence find the mean and variance.

Sol.
$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{0}^{\infty} e^{tx} \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(\frac{1}{2}-t)x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)}\right]_0^{\infty}$$

$$= \frac{-1}{2(\frac{1}{2}-t)}[0-1] = \frac{1}{1-2t}$$

$$M_X(t) = \frac{1}{1-2t} = (1-2t)^{-1}$$

$$= 1+2t+(2t)^2+\cdots$$

Mean $= E(X) = 2$

$E(X^2) = 8$

Variance $= E(X^2) - (E(X))^2$

$= 8 - 4$

$= 4$.

2) If $f(x) = kx^2e^{-2x}$, $x>0$ find i) value of k ii) Mean iii) Variance iv) r^{th} moment

Sol. Since the fn is pdf, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^{\infty} kx^2e^{-2x}dx = 1$$

$$k\left[x^2\left(\frac{e^{-2x}}{-2}\right) - (2x)\left(\frac{e^{-2x}}{4}\right) + (2)\left(\frac{e^{-2x}}{-8}\right)\right]_0^{\infty} = 1$$

$$k\left[0 - \left(0 - 0 - \frac{2}{8}\right)\right] = 1$$

$$\frac{k}{4} = 1 \Rightarrow k = 4.$$

r^{th} moment.

$$E(x^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^{\infty} x^r K x^2 e^{-2x}\,dx$$

$$= 4\int_0^{\infty} x^{r+2} e^{-2x}\,dx$$

put $2x = t$
$2dx = dt$

$$= 4\int_0^{\infty} \left(\frac{t}{2}\right)^{r+2} e^{-t}\,\frac{dt}{2}$$

$$= \frac{4}{2\cdot 2^{r+2}}\int_0^{\infty} t^{r+2} e^{-t}\,dt$$

$$E(x^r) = \frac{(r+2)!}{2^{r+1}}$$

$$\text{Mean} = E(x) = \frac{3!}{2^2} = \frac{6}{4} = \frac{3}{2}$$

$$E(x^2) = \frac{4!}{2^3} = \frac{24}{8} = 3$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= 3 - \frac{9}{4}$$

$$= \frac{3}{4}$$

3) Find the probability distribution of the total no. of heads obtained in four tosses of a balanced coin. Hence obtain the MGF of X, Mean and Variance of X.

Sol: probability dist. is

X :	0	1	2	3	4
$p(x)$:	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

$$M_x(t) = \sum_{x=0}^{4} e^{tx} p(x)$$

$$= \frac{1}{16} + e^{t}\frac{4}{16} + e^{2t}\frac{6}{16} + e^{3t}\cdot\frac{4}{16} + e^{4t}\cdot\frac{1}{16}$$

$$= \frac{1}{16}\left[1+4e^{t}+6e^{2t}+4e^{3t}+e^{4t}\right]$$

$$= \frac{1}{16}(1+e^{t})^{4}$$

$$\text{Mean} = E(x) = \left[M_x'(t)\right]_{t=0}$$

$$= \frac{1}{16}\left[4(1+e^{t})^{3}\cdot e^{t}\right]_{t=0}$$

$$= \frac{1}{16}\left[4(1+1)^{3}\cdot 1\right]$$

$$= \frac{32}{16} = 2.$$

$$E(x^2) = \left[M_x''(t)\right]_{t=0}$$

$$= \frac{1}{4}\left[(1+e^{t})^{3}\cdot e^{t} + e^{t}\cdot 3(1+e^{t})^{2}\cdot e^{t}\right]_{t=0}$$

$$= \frac{1}{4}\left[8+3(4)\right] = \frac{20}{4} = 5$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= 5-(2)^2$$

$$= 1.$$

Binomial distribution.

A random variable x is said to follow binomial distribution if it assumes only non-negative values and its pmf is given by

$$P(x=x) = p(x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2, \dots n$$

$$q = 1-p.$$

Note :- $\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} nC_x\, p^x q^{n-x}$

$$= (q+p)^n = 1.$$

M.G.F. of Binomial distribution

$$M_x(t) = E[e^{tx}] = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (pe^t)^x\, nC_x\, q^{n-x}$$

$$= nC_0\, q^n + nC_1 (pe^t) q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + nC_n (pe^t)^n$$

$$M_x(t) = (q+pe^t)^n$$

Mean and Variance of Binomial distribution

$$\text{Mean} = E(x) = [M_x'(t)]_{t=0}$$

$$= [n(q+pe^t)^{n-1} \cdot pe^t]_{t=0}$$

$M_x(t) = n(q+p)^{n-1} \cdot p$

$= n(1)^{n-1} p \qquad (\because q = 1-p$

$q+p = 1)$

$= np$

$E(x^2) = \left[M_x''(t) \right]_{t=0}$

$= np\left[(q+pe^t)^{n-1} \cdot e^t + e^t (n-1)(q+pe^t)^{n-2} \cdot pe^t \right]_{t=0}$

$= np\left[(q+p)^{n-1} + (n-1)(q+p)^{n-2} \cdot p \right]$

$= np\left[1 + (n-1)p \right]$

$= np + n^2p^2 - np^2$

Variance $= E(x^2) - (E(x))^2$

$= np + n^2p^2 - np^2 - (np)^2$

$= np - np^2$

$= np(1-p)$

$= npq$

Poisson distribution

A random variable x is said to follow poisson distribution if it assumes only non-negative values and its pmf is given by

$$P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x=0,1,2,\dots,\lambda>0$$

λ is known as the parameter of the Poisson distribution.

MGF of Poisson distribution

$$M_X(t) = E[e^{tx}] = \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^{t}}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \dots \infty\right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^{t}}$$

$$M_X(t) = e^{\lambda(e^{t}-1)}$$

Mean and Variance of Poisson distribution

$$\text{Mean} = E(X) = [M_X'(t)]_{t=0} = \left[e^{-\lambda} \cdot e^{\lambda e^{t}} \cdot \lambda e^{t}\right]_{t=0}$$

$$= e^{-\lambda} e^{\lambda} \cdot \lambda$$

$$= \lambda.$$

$$E(x^2) = [M_x''(t)]_{t=0}$$

$$= \lambda e^{-\lambda}[e^{\lambda e^t} \cdot e^t + e^t \cdot e^{\lambda e^t} \cdot \lambda e^t]_{t=0}$$

$$= \lambda e^{-\lambda}[e^{\lambda} + e^{\lambda} \cdot \lambda]$$

$$= \lambda + \lambda^2$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \lambda + \lambda^2 - \lambda^2$$

$$= \lambda .$$

Geometric Distribution

The pmf for geometric distribution is

$$p(x = x) = p(x)$$

$$= (1-p)^{x-1} p$$

$$= q^{x-1} p, \quad x = 1, 2, 3 \text{ - - - - .}$$

(or)

$$p(x) = q^{x} p, \quad x = 0, 1, 2, 3, \text{ - - - - - .}$$

MGF of Geometric distribution

$$M_x(t) = E[e^{tx}]$$

$$= \sum e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p e^{t} \cdot e^{-t}$$

$$M_X(t) = \sum_{x=1}^{\infty} e^{t(x-1)} q^{x-1} pe^t$$

$$= \sum_{x=1}^{\infty} pe^t (qe^t)^{x-1}$$

$$= pe^t [1 + qe^t + (qe^t)^2 + \cdots\cdots]$$

$$= pe^t (1-qe^t)^{-1}$$

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

Mean and Variance of Geometric distribution

$$\text{Mean} = E(X) = [M_X'(t)]_{t=0}$$

$$= \left[\frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

$$E(x^2) = [M_x''(t)]_{t=0} = \left[\frac{(1-qe^t)^2 pe^t - pe^t 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]_{t=0}$$

$$= \left[\frac{(1-q)^2 p - p2(1-q)(-q)}{(1-q)^4}\right]$$

$$= \frac{p(1-q) + 2pq}{(1-q)^3} = \frac{p+pq}{(1-q)^3}.$$

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$$E(x^2) = \frac{p(1+q)}{p^3} = \frac{p(1+1-p)}{p^3}$$
$$= \frac{2-p}{p^2}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$
$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$
$$= \frac{2-p-1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Memoryless property of Geometric distribution

If X is a discrete RV following a geometric distribution then $p(x>m+n/x>m) = p(x>n)$.

Proof:- Since X follows a geometric distribution,

$$p(x=x) = p(x) = pq^{x-1}, \quad x = 1, 2, 3, \ldots \infty$$

$$p(x>k) = \sum_{x=k+1}^{\infty} pq^{x-1} = pq^k + pq^{k+1} + pq^{k+2} + \ldots$$
$$= pq^k(1+q+q^2+\ldots)$$
$$= pq^k(1-q)^{-1} = \frac{pq^k}{1-q} = \frac{pq^k}{p} = q^k.$$

Now, $p(x>m+n/x>m) = \frac{p(x>m+n \cap x>m)}{p(x>m)}$

$$= \frac{p(x>m+n)}{p(x>m)} = \frac{q^{m+n}}{q^m} = q^n$$
$$= p(x>n)$$

Uniform distribution (Continuous or Rectangular distribution)

pdf for uniform distribution is

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$

$$= 0 \quad \text{otherwise}$$

MGF of Uniform distribution

$$M_X(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$M_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

Mean and Variance of Uniform distribution

$$M_X(t) = \frac{1}{(b-a)t}\left[\left(1 + \frac{bt}{1!} + \frac{b^2t^2}{2!} + \cdots\right) - \left(1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \cdots\right)\right]$$

$$= \frac{1}{(b-a)t}\left[t(b-a) + \frac{t^2}{2}(b^2-a^2) + \frac{t^3}{6}(b^3-a^3) + \cdots\right]$$

$$= 1 + \frac{t}{2}(b+a) + \frac{t^2}{6}(b^2+ab+a^2) + \cdots\cdots$$

$$\text{Mean} = E(x) = \text{Coeff of } t = \frac{b+a}{2}$$

$$E(x^2) = \text{Coeff of } \frac{t^2}{2!} = \frac{2}{6}(b^2+ab+a^2)$$

$$= \frac{1}{3}(b^2+ab+a^2)$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

Exponential distribution

Pdf for exponential distribution is

$$f(x) = c e^{-cx}, \quad x \geq 0, \; c > 0$$

$$= 0 \quad \text{otherwise}$$

MGF of Exponential distribution.

$$M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} . c e^{-cx} dx$$

$$= c\int_0^{\infty} e^{-(c-t)x} dx$$

$$= c\left[\frac{e^{-(c-t)x}}{-(c-t)}\right]_0^{\infty}$$

$$= \frac{c}{-(c-t)}[0-1]$$

$$= \frac{c}{c-t}$$

Mean and Variance of Exponential distribution

$$M_X(t) = \frac{c}{c-t} = \frac{c}{c\left(1-\frac{t}{c}\right)} = \left(1-\frac{t}{c}\right)^{-1}$$

$$= 1 + \frac{t}{c} + \frac{t^2}{c^2} + \cdots$$

$$\text{Mean} = E(x) = \text{Coeff of } t = \frac{1}{c}$$

$$E(x^2) = \text{Coeff of } \frac{t^2}{2!} = \frac{2}{c^2}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \frac{2}{c^2} - \left(\frac{1}{c}\right)^2$$

$$= \frac{1}{c^2}$$

Memoryless property of Exponential distribution

If x is continuous RV following exponential distribution then $p(x > s+t \mid x > s) = p(x > t)$ for any $s, t > 0$.

Proof:- pdf of exponential distribution is

$$f(x) = c\,e^{-cx}, \quad x \geq 0, \quad c > 0.$$

$$P(x > k) = \int_k^\infty f(x)\,dx = \int_k^\infty c\,e^{-cx}\,dx = c\left[\frac{e^{-cx}}{-c}\right]_k^\infty$$

$$= -\left[0 - e^{-ck}\right] = e^{-ck}$$

$$\text{Now, } p(x > s+t \mid x > s) = \frac{p(x > s+t \cap x > s)}{p(x > s)}$$

$$= \frac{p(x > s+t)}{p(x > s)} = \frac{e^{-c(s+t)}}{e^{-cs}} = e^{-ct} = p(x > t)$$

Normal distribution (or) Gaussian distribution

Pdf of $f(x)$ with parameter μ and σ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

Symbollically x follows $N(\mu, \sigma)$

Note:- When we use the transformation $z = \frac{x-\mu}{\sigma}$, the normal distribution with mean μ and S.D σ. (ie) $N(\mu, \sigma)$ is converted into standard normal distribution $N(0,1)$ with mean 0 and S.D. 1.

In this case the density fn is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

MGF of $N(0,1)$

$$M_z(t) = E[e^{tz}] = \int_{-\infty}^{\infty} e^{tz} f(z)\,dz$$

$$= \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2-2tz]} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-t)^2-t^2]} dz$$

$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} dz$$

$$M_z(t) = \frac{e^{+t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z-t}{\sqrt{2}}\right)^2} dz$$

$$\boxed{\begin{array}{c} \text{put } \frac{z-t}{\sqrt{2}} = u \\ dz = \sqrt{2}\, du \\ \hline \because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \end{array}}$$

$$= \frac{e^{+t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \cdot \sqrt{2}\, du$$

$$= \frac{e^{+t^2/2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$M_z(t) = \frac{e^{+t^2/2}}{\sqrt{\pi}} \sqrt{\pi} \quad =$$

$$M_z(t) = e^{t^2/2}$$

MGF of $N(\mu, \sigma)$

We have $z = \frac{x-\mu}{\sigma} \Rightarrow x = z\sigma + \mu$

$$M_X(t) = E[e^{tx}]$$

$$= E[e^{t(z\sigma+\mu)}]$$

$$= E[e^{tz\sigma} \cdot e^{t\mu}]$$

$$= e^{t\mu} E[e^{(t\sigma)z}]$$

$$= e^{t\mu} M_z(t\sigma)$$

$$M_X(t) = e^{t\mu} \cdot e^{t^2\sigma^2/2}$$

Mean and Variance of $N(\mu,\sigma)$

$$\text{Mean} = \left[M_x'(t)\right]_{t=0}$$

$$E(x) = \left[e^{t\mu}\cdot e^{t^2\sigma^2/2}\cdot\frac{\cancel{2}t\sigma^2}{\cancel{2}} + e^{t^2\sigma^2/2}\cdot e^{t\mu}\cdot\mu\right]_{t=0}$$

$$= 0+\mu$$

$$= \mu.$$

$$E(x^2) = \left[M_x''(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{e^{t\mu}\cdot e^{t^2\sigma^2/2}(t\sigma^2+\mu)\right\}\right]_{t=0}$$

$$= \Big[e^{t\mu}e^{t^2\sigma^2/2}(\sigma^2) + e^{t\mu}(t\sigma^2+\mu)e^{t^2\sigma^2/2}\cdot\frac{2t\sigma^2}{\cancel{2}}$$

$$+ e^{t^2\sigma^2/2}(t\sigma^2+\mu)\cdot e^{t\mu}\cdot\mu\Big]_{t=0}$$

$$= \sigma^2+0+\mu^2$$

$$= \sigma^2+\mu^2$$

$$\text{Variance} = E(x^2)-(E(x))^2$$

$$= \sigma^2+\mu^2-\mu^2$$

$$= \sigma^2$$

Note:- For normal distribution,

mean = median = mode $= \mu$.

Problems.

1) In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that i) all are good bulbs ii) atmost there are 3 defective bulbs iii) exactly there are 3 defective bulbs.

Sol. Here $n=20$, $p=\frac{10}{100}=0.1$

$q=1-p=1-0.1=0.9$

X follows binomial distribution.

i) $p(\text{all are good bulbs}) = p(x=0)$

$= 20C_0 (0.1)^0 (0.9)^{20}$

$= 0.1216$

ii) $p(x \leq 3) = p(x=0)+p(x=1)+p(x=2)+p(x=3)$

$= 20C_0 (0.1)^0 (0.9)^{20} + 20C_1 (0.1)(0.9)^{19}$

$+ 20C_2 (0.1)^2 (0.9)^{18} + 20C_3 (0.1)^3 (0.9)^{17}$

$= 0.8671$

iii) $p(x=3) = 20C_3 (0.1)^3 (0.9)^{17}$

$= 0.1901$

2) The no. of monthly breakdown of a computer is a RV having a Poisson distribution with mean equal to 1.8 Find the probability that ~~[illegible]~~ this computer will function for a month

i) without breakdown

ii) with only one breakdown

iii) with atleast one breakdown.

Sol: Let X denote the no. of breakdowns of the computer in a month.

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{given } \lambda = 1.8$$

i) $P(\text{without a breakdown}) = P(X=0)$

$$= \frac{e^{-1.8}(1.8)^{0}}{0!}$$

$$= e^{-1.8} = 0.1653$$

ii) $P(\text{with only one breakdown}) = P(X=1)$

$$= \frac{e^{-1.8}(1.8)^{1}}{1!}$$

$$= 0.2975$$

iii) $P(\text{with atleast one breakdown}) = P(X \geq 1)$

$$= 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - 0.1653$$

$$= 0.8347$$

3) Suppose that a trainee soldier shots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.7

i) What is the probability that the target would be hit on tenth attempt?

ii) What is the probability that it takes him less than 4 shots?

iii) What is the probability that it takes him an even no. of shots?

iv) What is the average no. of shots needed to hit the target?

Sol. X follows geometric distribution.

$$p = 0.7, \quad q = 1 - 0.7 = 0.3$$

$$P(X = x) = q^{x-1} p, \quad x = 1, 2 \text{----} \infty$$

i) $P(X=10) = (0.3)^9 (0.7) = 0.000014$

ii) $P(X<4) = \sum_{x=1}^{3} q^{x-1} p = q^0 p + q^1 p + q^2 p$

$$= 0.7 + (0.3)(0.7) + (0.3)^2 (0.7)$$

$$= 0.973$$

iii) $P(\text{even no. of shots}) = P(X=2) + P(X=4) + P(X=6) + \text{-----}$

$$= (0.3)(0.7)\left[1 + (0.3)^2 + (0.3)^4 + \text{----}\right]$$

$$= (0.3)(0.7)(1 - 0.09)^{-1} = 0.2308$$

iv) Average no. of shots $= E(X) = \frac{1}{p} = \frac{1}{0.7}$

$$= 1.4286$$

4) If x is uniformly distributed over $(-\alpha, \alpha)$, $\alpha > 0$

Find α so that i) $P(X>1) = \frac{1}{3}$

ii) $P(|X|<1) = P(|X|>1)$

Sol. Since x is uniformly distributed over $(-\alpha, \alpha)$

we have $f(x) = \frac{1}{\alpha+\alpha} = \frac{1}{2\alpha}$, $-\alpha < x < \alpha$

i) $P(X>1) = \frac{1}{3}$

$$\int_1^\alpha f(x)\,dx = \frac{1}{3}$$

$$\int_1^\alpha \frac{1}{2\alpha}\,dx = \frac{1}{3} \Rightarrow \frac{1}{2\alpha}\left[x\right]_1^\alpha = \frac{1}{3}$$

$$\Rightarrow \frac{1}{2\alpha}(\alpha-1) = \frac{1}{3}$$

$$3\alpha - 3 = 2\alpha$$

$$\alpha = 3.$$

ii) $P(|X|<1) = P(|X|>1)$

$= 1 - P(|X| \leq 1)$

$$2P(|X|<1) = 1$$

$$2P(-1<X<1) = 1$$

$$2\int_{-1}^{1} \frac{1}{2\alpha}\,dx = 1$$

$$\frac{1}{\alpha}\left[x\right]_{-1}^{1} = 1$$

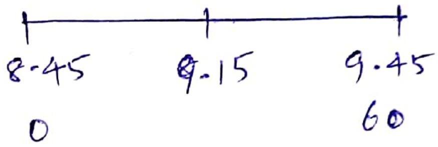
$$\frac{1}{\alpha}(1+1) = 1$$

$$\alpha = 2.$$

5) Starting at 5.00 am every half hour there is a flight from San Francisco airport to Los Angeles airport. Suppose that none of these planes is completely sold out and that they always have room for passengers. A person who wants to fly to L.A arrive at the airport at a random time between 8.45am and 9.45am Find the probability that she waits i) atmost 10 minutes ii) atleast 15 minutes.

Sol. Let X be the uniform RV over $(0, 60)$

Then the pdf is $f(x) = \frac{1}{60}, 0 < x < 60$



i) P(she waits atmost 10 min.)

= P(she arrives between 9.05 and 9.15 or 9.35 and 9.45)

$= P(20 < X < 30) + P(50 < X < 60)$

$$= \int_{20}^{30} \frac{1}{60}\,dx + \int_{50}^{60} \frac{1}{60}\,dx = \frac{10}{60} + \frac{10}{60} = \frac{1}{3}.$$

ii) P(she waits atleast 15 min.)

= P(she arrives between 8.45 and 9.00 or 9.15 and 9.30)

$= P(0 < X < 15) + P(30 < X < 45)$

$$= \int_{0}^{15} \frac{1}{60}\,dx + \int_{30}^{45} \frac{1}{60}\,dx = \frac{15}{60} + \frac{15}{60} = \frac{1}{2}.$$

6) The time (in hours) required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$.

i) What is the probability that the repair time exceeds 2h?

ii) What is the conditional probability that a repair takes atleast 11h given that its direction exceeds 8h?

Sol.

$$f(x) = \cancel{\lambda} e^{-\cancel{\lambda} x}, \quad x > 0$$

$$= \frac{1}{2} e^{-x/2}, \quad x > 0$$

i) $$p(X>2) = \int_2^\infty f(x)\,dx = \int_2^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^\infty$$

$$= -\left[0 - e^{-1}\right] = \frac{1}{e} = 0.3679$$

ii) $$p(X>11/X>8) = p(X>3)$$

$$= \int_3^\infty \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_3^\infty$$

$$= -\left[0 - e^{-3/2}\right] = e^{-3/2}$$

$$= 0.2231.$$

7) If a continuous RV, X follows uniform distribution in the interval (0, 2) and a continuous RV, Y follows exponential distribution with parameter λ Find λ such that $P(X<1) = P(Y<1)$

Sol. Since X follows uniform distribution over (0, 2)

$$f(x) = \frac{1}{2}, \quad 0<x<2$$

Y follows exponential distribution

$$f(y) = \lambda e^{-\lambda y}, \quad y>0$$

Given $P(X<1) = P(Y<1)$

$$\int_0^1 f(x)dx = \int_0^1 f(y)dy$$

$$\int_0^1 \frac{1}{2} dx = \int_0^1 \lambda e^{-\lambda y} dy$$

$$\frac{1}{2}[x]_0^1 = \lambda \left[\frac{e^{-\lambda y}}{-\lambda}\right]_0^1$$

$$\frac{1}{2} = -[e^{-\lambda} - 1]$$

$$e^{-\lambda} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$e^{\lambda} = 2$$

$$\lambda = \log_e 2$$

8) An electrical firm manufacturers light bulbs that have a life before burn out, that is normally distributed with mean equal to 800 hrs and a S.D. of 40 hrs. Find i) the probability that a bulb burns more than 834 hrs ii) the probability that bulb burns between 778 and 834 hrs.

Sol. Given mean $= \mu = 800$, S.D. $\sigma = 40$

i) $p(X > 834) = ?$

Standardize, $z = \frac{x-\mu}{\sigma} = \frac{x-800}{40}$

At $x = 834$, $z = \frac{834-800}{40} = 0.85$

$\therefore p(x > 834) = p(z > 0.85)$

$= 0.1977$

ii) $p(778 < X < 834) = ?$

Standardize, $z = \frac{x-\mu}{\sigma} = \frac{x-800}{40}$

At $x = 778$, $z = \frac{778-800}{40} = -0.55$

$p(778 < x < 834) = p(-0.55 < z < 0.85)$

$= p(-0.55 < z < 0) + p(0 < z < 0.85)$

$= p(0 < z < 0.55) + p(0 < z < 0.85)$

$= 0.2088 + 0.3023$

$= 0.5111$

9) If the actual amount of instant Coffee which a filling machine puts into 6- ounce jars is a RV having a normal distribution with S.D = 0.05 ounce and if only 3% of the jars are to contain less than 6 ounces of Coffee, what must be the mean fill of these jars?

Sol. Given $p(x<6) = 3\% = 0.03$

$$z = \frac{x-\mu}{\sigma} = \frac{x-\mu}{0.05}$$

At $x=6$, $z = \frac{6-\mu}{0.05} = -z_1$

$\therefore p(x<6) = 0.03$ becomes

$$p(z<-z_1) = 0.03$$

$$0.5 - p(0<z<z_1) = 0.03$$

$$p(0<z<z_1) = 0.47$$

$$z_1 = 1.808 \text{ [From the table]}$$

$$\Rightarrow \frac{\mu-6}{0.05} = 1.808$$

$$\mu = 6+0.05(1.808)$$

$$\mu = 6.094 \text{ ounces.}$$

10) In a class of 50, the average mark of students in a subject is 48 and S.D is 24. Find the no. of students who get i) above 50 ii) between 35 to 50.

Sol. $\mu = 48$, $S.D = 24$

i) $p(x > 50) = ?$

$$z = \frac{x-\mu}{\sigma} = \frac{x-48}{24}$$

At $x = 50$, $z = \frac{50-48}{24} = \frac{1}{12} = 0.08$

$$p(x > 50) = p(z > 0.08)$$
$$= 0.4681$$

$\therefore$ No. of students who got above 50 is 0.4681×50
$= 23$.

ii) $p(35 < x < 50) = ?$

$$z = \frac{x-\mu}{\sigma} = \frac{x-48}{24}$$

When $x = 35$, $z = -0.54$

$x = 50$, $z = 0.08$

$$p(35 < x < 50) = p(-0.54 < z < 0.08)$$
$$= p(-0.54 < z < 0) + p(0 < z < 0.08)$$
$$= p(0 < z < 0.54) + p(0 < z < 0.08)$$
$$= 0.2054 + 0.0319$$
$$= 0.2373$$

$\therefore$ No. of students who got between 35 and 50 is
$0.2373 \times 50 = 13$

RANDOM PROCESSES.

A random variable is a rule (or fn) that assigns a real no. to every outcome of a random experiment, while a random process is a rule that assigns a time fn to every outcome of a random experiment.

Definition: A random process is a collection of random variables $\{x(s,t)\}$ that are fn's of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set)

The set of possible values of any individual member of the random process is called state space. Any individual member itself is called a sample fn.

If the parameter t is discrete, the random process will be denoted by $\{x(n)\}$ or $\{x_n\}$.

If the parameter t is continuous, the random process will be denoted by $\{x(t)\}$ or $\{x_t\}$.

Classification of Random Processes (Stochastic process)

The random processes can be classified into 4 types

i) If both s and t are discrete, the random process is called discrete random sequence.

Eg: Tossing 3 coins 50 times.

ii) If s is continuous and t is discrete, the random process is called a continuous random sequence.

Eg: Heights of students at different ages.

iii) If s is discrete and t is continuous, the random process is called a discrete random process.

Eg: No. of telephone calls received in a telephone exchange in $(0,t)$.

iv) If both s and t are continuous, the random process is called continuous random process.

Eg: Temperature in $(0,t)$.

Stationary processes

If certain probability distribution or averages do not depend on t, then the random process $\{x(t)\}$ is called stationary.

A random process that is not stationary in any sense is called an evolutionary process.

Average values of Random Processes.

Random Processes can be described in terms of averages or expected values, mostly derived from the first and second order distributions of $\{x(t)\}$.

i) Mean of the process $\{x(t)\}$ is the expected value of a typical member $\{x(t)\}$ of the process.

ie) $\mu(t) = E\{x(t)\}$.

ii) Autocorrelation of the process $\{X(t)\}$ denoted by $R_{XX}(t_1, t_2)$ or $R_X(t_1, t_2)$ or $R(t_1, t_2)$ is the expected value of the product of any two numbers $X(t_1)$ and $X(t_2)$ of the process. (ie) $R(t_1, t_2) = E[X(t_1) . X(t_2)]$

Definition.

1) A random process is called a strongly stationary process or strict sense stationary process (SSS process) if all its finite dimensional distributions are invariant under translation of time parameter.

(ie) $E\{X(t)\} = \mu$ = a constant

and $R(t_1, t_2)$ is a fn. of $(t_1 - t_2)$.

2) Two real valued random process $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the strict sense, if the joint distribution of $X(t)$ and $Y(t)$ are invariant under translation of time.

3) A random process $\{X(t)\}$ is called weakly stationary process or wide-sense stationary process (WSS process) if its mean is a constant and the autocorrelation depends only on the time difference.

4) Two random process $\{X(t)\}$ and $\{Y(t)\}$ are said to be jointly stationary in the wide sense, if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a fn. of $(t_1 - t_2)$ only.

Problems.

1) Examine whether the Poisson process $\{X(t)\}$ given by the probability law $P\{X(t)=r\} = e^{-\lambda t}\frac{(\lambda t)^r}{r!}, r=0,1,2\ldots$ is covariance stationary.

Sol. The probability distribution of $X(t)$ is a Poisson distribution with parameter λt.

$$\therefore E\{X(t)\} = \lambda t \neq \text{a constant}$$

$\therefore$ Poisson process is not covariance stationary.

---

2) Consider the random process $X(t) = \cos(t+\phi)$, where ϕ is a RV with density fn. $f(\phi) = \frac{1}{\pi}, -\frac{\pi}{2} < \phi < \frac{\pi}{2}$ check whether the process is stationary or not.

Sol. Given $X(t) = \cos(t+\phi)$, $f(\phi) = \frac{1}{\pi}$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

$$E[X(t)] = \int_{-\pi/2}^{\pi/2} X(t) f(\phi)\, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \cos(t+\phi) \cdot \frac{1}{\pi}\, d\phi$$

$$= \frac{1}{\pi}\Big[\sin(t+\phi)\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}+t\right) - \sin\left(-\frac{\pi}{2}+t\right)\right]$$

$$= \frac{1}{\pi}\left[\cos t + \sin\left(\frac{\pi}{2}-t\right)\right]$$

$$E[X(t)] = \frac{1}{\pi}(\cos t + \cos t)$$

$$= \frac{2\cos t}{\pi} = \text{a fn. of } t$$

$\therefore X(t)$ is not a stationary process.

---

3) The process $\{X(t)\}$ whose probability distribution under certain conditions is given by

$$P\{X(t)=n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n=1,2,\text{----}$$

$$= \frac{at}{1+at}, \quad n=0.$$

Show that it is not stationary (or evolutionary)?

Sol: The probability dist. of $X(t)$ is

$X(t)=n$	:	0	1	2	3	------
P_n	:	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	----

$$E[X(t)] = \sum n P_n$$

$$= 0 + \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \text{-----}$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2at}{1+at} + \frac{3(at)^2}{(1+at)^2} + \text{----}\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}(1+at)^2$$

$$E[X(t)] = 1$$

$$E[X^2(t)] = \sum n^2 p_n$$

$$= 0 + \frac{1}{(1+at)^2} + \frac{4at}{(1+at)^3} + \frac{9(at)^2}{(1+at)^4} + \frac{16(at)^3}{(1+at)^5} + \text{-----}$$

$$= \left[\frac{1}{(1+at)^2} + \frac{3at}{(1+at)^3} + \frac{6(at)^2}{(1+at)^4} + \frac{10(at)^3}{(1+at)^5} + \text{----}\right]$$

$$+ \left[\frac{at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \frac{6(at)^3}{(1+at)^5} + \text{----}\right]$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{3at}{1+at} + \frac{6(at)^2}{(1+at)^2} + \frac{10(at)^3}{(1+at)^3} + \text{----}\right]$$

$$+ \frac{at}{(1+at)^3}\left[1 + \frac{3at}{1+at} + \frac{6(at)^2}{(1+at)^2} + \text{----}\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-3} + \frac{at}{(1+at)^3}\left[1 - \frac{at}{1+at}\right]^{-3}$$

$$= \frac{1}{(1+at)^2}\left(\frac{1}{1+at}\right)^{-3} + \frac{at}{(1+at)^3}(1+at)^3$$

$$= 1+at+at$$

$$= 1+2at$$

$$\text{Var}\{X(t)\} = E[X^2(t)] - [E\{X(t)\}]^2$$

$$= 1+2at-1$$

$$= 2at$$

If $X(t)$ is a stationary process, $E[X(t)]$ and $\text{Var}\{X(t)\}$ are constants.

Since $\text{Var}\{X(t)\}$ is a fn. of t, the given process is not stationary.

4) Show that the random process $X(t) = A\cos(w_0 t+\theta)$ is WSS process if A and w_0 are constants and θ is uniformly distributed RV in $(0, 2\pi)$.

Sol. Since θ is uniformly distributed in $(0, 2\pi)$

$$f(\theta) = \frac{1}{2\pi}, \quad 0 < \theta < 2\pi$$

$$E[X(t)] = \int_0^{2\pi} X(t) f(\theta)\, d\theta$$

$$= \int_0^{2\pi} A\cos(w_0 t+\theta) \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi}\Big[\sin(w_0 t+\theta)\Big]_0^{2\pi}$$

$$= \frac{A}{2\pi}\Big[\sin w_0 t - \sin w_0 t\Big]$$

$= 0$ = a constant

$$R(t_1,t_2) = E[X(t_1).X(t_2)]$$

$$= E\Big[A\cos(w_0 t_1+\theta). A\cos(w_0 t_2+\theta)\Big]$$

$$= E\Big[A^2 \cdot \frac{1}{2}\Big\{\cos(w_0 t_1+\theta+w_0 t_2+\theta) + \cos(w_0 t_1+\theta - w_0 t_2 - \theta)\Big\}\Big]$$

$$= \frac{A^2}{2} E\Big[\cos\{(t_1+t_2)w_0+2\theta\} + \cos\{(t_1-t_2)w_0\}\Big]$$

$$= \frac{A^2}{2}\int_0^{2\pi}\Big[\cos\{(t_1+t_2)w_0+2\theta\} + \cos\{(t_1-t_2)w_0\}\Big] \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin\{(t_1+t_2)w_0+2\theta\}}{2} + \theta\cos\{(t_1-t_2)w_0\}\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[\left(\frac{\sin\{(t_1+t_2)w_0+4\pi\}}{2} + 2\pi\cos\{(t_1-t_2)w_0\}\right) - \left(\frac{\sin\{(t_1+t_2)w_0+0\}}{2} + 0\right)\right]$$

$$R(t_1,t_2) = \frac{A^2}{4\pi}\left[\cancel{\frac{\sin(t_1+t_2)w_0}{2}} + 2\pi\cos(t_1-t_2)w_0 - \cancel{\frac{\sin(t_1+t_2)w_0}{2}}\right]$$

$$= \frac{A^2}{2}\cos w_0(t_1-t_2)$$

$$= \text{a fn. of } (t_1-t_2)$$

$\therefore \{x(t)\}$ is a WSS process.

5) Given a RV Y with $\phi(\omega) = E[e^{i\omega Y}] = E[\cos\omega Y + i\sin\omega Y]$ and a random process defined by $x(t) = \cos(\lambda t + Y)$. Show that $\{x(t)\}$ is stationary in the wide sense if $\phi(1) = \phi(2) = 0$.

Sol.

$$E[x(t)] = E[\cos(\lambda t + Y)]$$

$$= E[\cos\lambda t\cos Y - \sin\lambda t\sin Y]$$

$$= \cos\lambda t\, E(\cos Y) - \sin\lambda t\, E(\sin Y) \quad ①$$

Given $\phi(\omega) = E(\cos\omega Y + i\sin\omega Y)$

When $\phi(1) = 0$, we get $0 = E(\cos Y + i\sin Y)$

$$\Rightarrow E(\cos Y) = 0,\ E(\sin Y) = 0.$$

Now eqn. ① becomes

$$E[x(t)] = \cos\lambda t\cdot 0 - \sin\lambda t\cdot 0$$

$$= 0 = \text{a constant}$$

$$R(t_1,t_2) = E[x(t_1)\cdot x(t_2)]$$

$$= E[\cos(\lambda t_1 + Y)\cdot\cos(\lambda t_2 + Y)]$$

$$= E[(\cos\lambda t_1\cos Y - \sin\lambda t_1\sin Y)(\cos\lambda t_2\cos Y - \sin\lambda t_2\sin Y)]$$

$$R(t_1, t_2) = E\left[\cos\lambda t_1 \cos\lambda t_2 \cos^2 y - \sin\lambda t_1 \cos\lambda t_2 \sin y \cos y - \cos\lambda t_1 \sin\lambda t_2 \sin y \cos y + \sin\lambda t_1 \sin\lambda t_2 \sin^2 y\right]$$

$$= \cos\lambda t_1 \cos\lambda t_2 \, E(\cos^2 y) - \{\sin\lambda t_1 \cos\lambda t_2 + \cos\lambda t_1 \sin\lambda t_2\} \cdot E(\sin y \cos y) + \sin\lambda t_1 \sin\lambda t_2 \, E(\sin^2 y)$$

$$= \cos\lambda t_1 \cos\lambda t_2 \, E\left(\frac{1+\cos 2Y}{2}\right) - \sin\lambda(t_1+t_2) \, E\left(\frac{\sin 2Y}{2}\right) + \sin\lambda t_1 \sin\lambda t_2 \, E\left(\frac{1-\cos 2Y}{2}\right) \quad \text{②}$$

Now $\phi(\omega) = E(\cos wy + i \sin wy)$

When $\phi(2) = 0$, we get $0 = E(\cos 2y + i \sin 2Y)$

$$\Rightarrow E(\cos 2y) = 0, \quad E(\sin 2y) = 0$$

$$\text{②} \Rightarrow R(t_1, t_2) = \cos\lambda t_1 \cos\lambda t_2 \, E(\tfrac{1}{2}) - 0 + \sin\lambda t_1 \sin\lambda t_2 \, E(\tfrac{1}{2})$$

$$= \frac{1}{2}\left[\cos\lambda t_1 \cos\lambda t_2 + \sin\lambda t_1 \sin\lambda t_2\right]$$

$$= \frac{1}{2}\cos\lambda(t_1 - t_2)$$

$$= \text{a fn of } (t_1 - t_2)$$

$\therefore \{X(t)\}$ is a WSS process.

6) S.T. the process $X(t) = A\cos\lambda t + B\sin\lambda t$ is WSS if $E(A) = E(B) = 0$, $E(A^2) = E(B^2)$ and $E(AB) = 0$ (or) WSS process with zero mean, same variance and independent RV & uncorrelated.

Sol.
$$E[x(t)] = E[A\cos\lambda t + B\sin\lambda t]$$
$$= \cos\lambda t\, E(A) + \sin\lambda t\, E(B)$$
$$= \cos\lambda t \cdot 0 + \sin\lambda t \cdot 0$$
$$= 0$$
= a constant

$$R(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$$
$$= E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$$
$$= E[A^2\cos\lambda t_1 \cos\lambda t_2 + AB\cos\lambda t_1 \sin\lambda t_2 + AB\sin\lambda t_1 \cos\lambda t_2 + B^2\sin\lambda t_1 \sin\lambda t_2]$$
$$= \cos\lambda t_1 \cos\lambda t_2\, E(A^2) + \sin\lambda(t_1 + t_2)\, E(AB) + \sin\lambda t_1 \sin\lambda t_2\, E(B^2)$$

Given $E(A^2) = E(B^2) = K$ (say) and $E(AB) = 0$.

$$R(t_1, t_2) = K\cos\lambda t_1 \cos\lambda t_2 + 0 + K\sin\lambda t_1 \sin\lambda t_2$$
$$= K\cos\lambda(t_1 - t_2)$$
= a fn. of $(t_1 - t_2)$

$\therefore \{x(t)\}$ is a WSS process.

7) Two random process $X(t)$ and $Y(t)$ are defined by $X(t) = A\cos w_0 t + B\sin w_0 t$ and $Y(t) = B\cos w_0 t - A\sin w_0 t$ Show that $X(t)$ and $Y(t)$ are jointly WSS, if A and B are uncorrelated RV's with zero means and the same variances and w_0 is a constant.

Sol.

$$E[X(t)] = E[A\cos w_0 t + B\sin w_0 t]$$

$$= E(A)\cos w_0 t + E(B)\sin w_0 t$$

$$= 0 \qquad [\because E(A) = E(B) = 0]$$

$$= \text{a constant}$$

$$R_{XX}(t_1,t_2) = E[X(t_1)\cdot X(t_2)]$$

$$= E[(A\cos w_0 t_1 + B\sin w_0 t_1)(A\cos w_0 t_2 + B\sin w_0 t_2)]$$

$$= E[A^2\cos w_0 t_1 \cos w_0 t_2 + AB\cos w_0 t_1 \sin w_0 t_2 + AB\sin w_0 t_1 \cos w_0 t_2 + B^2\sin w_0 t_1 \sin w_0 t_2]$$

$$= E(A^2)\cos w_0 t_1 \cos w_0 t_2 + E(AB)\sin w_0(t_1+t_2) + E(B^2)\sin w_0 t_1 \sin w_0 t_2$$

$$= k[\cos w_0 t_1 \cos w_0 t_2 + \sin w_0 t_1 \sin w_0 t_2] + 0$$

$$= k\cos w_0(t_1 - t_2)$$

$\therefore \{X(t)\}$ is a WSS process.

$$E[Y(t)] = E[B\cos w_0 t - A\sin w_0 t]$$

$$= E(B)\cos w_0 t - E(A)\sin w_0 t$$

$$= 0$$

$$= \text{a constant}$$

$$R_{yy}(t_1,t_2) = E[Y(t_1)Y(t_2)]$$

$$= E[(B\cos w_0 t_1 - A\sin w_0 t_1)(B\cos w_0 t_2 - A\sin w_0 t_2)]$$

$$= E[B^2 \cos w_0 t_1 \cos w_0 t_2 - AB\cos w_0 t_1 \sin w_0 t_2$$

$$- AB\sin w_0 t_1 \cos w_0 t_2 + A^2 \sin w_0 t_1 \sin w_0 t_2]$$

$$= K[\cos w_0 t_1 \cos w_0 t_2 + \sin w_0 t_1 \sin w_0 t_2] - 0$$

$$= K\cos w_0 (t_1 - t_2)$$

$\therefore Y(t)$ is WSS process.

Now, $R_{xy}(t_1,t_2) = E[X(t_1)\cdot Y(t_2)]$

$$= E[(A\cos w_0 t_1 + B\sin w_0 t_1)(B\cos w_0 t_2 - A\sin w_0 t_2)]$$

$$= E(AB)\cos w_0 t_1 \cos w_0 t_2 - E(A^2)\cos w_0 t_1 \sin w_0 t_2$$

$$+ E(B^2)\sin w_0 t_1 \cos w_0 t_2 - E(AB)\sin w_0 t_1 \sin w_0 t_2$$

$$= K[\sin w_0 t_1 \cos w_0 t_2 - \cos w_0 t_1 \sin w_0 t_2]$$

$$= K\sin w_0 (t_1 - t_2)$$

$= $ a fn. of $t_1 - t_2$

Hence $\{X(t)\}$ and $\{Y(t)\}$ are jointly WSS process.

Markov Process

If the future behaviour of a process depends only on the present state, but not on the past, the process is a Markov process. A discrete parameter Markov process is called a Markov chain.

Markov Chain.

If for all n,

$$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots X_0 = a_0]$$
$$= P[X_n = a_n / X_{n-1} = a_{n-1}]$$

then the process $\{X_n\}$, $n = 0, 1, 2, \dots$ is called a Markov chain.

$(a_1, a_2 \dots a_n)$ are called the states of the Markov chain.

The Condition probability $P[X_n = a_j / X_{n-1} = a_i]$ is called the one-step transition probability from state a_i to state a_j at the n^{th} step (trial) and is denoted by $P_{ij}(n-1, n)$ forms a matrix $[P_{ij}]$.

Classification of states of a Markov chain.

If every state can be reached from every other state, then the Markov chain is said to be irreducible. Otherwise, the ~~state~~ chain is said to be reducible or non irreducible.

The period d_i is defined as the GCD of all m where m is the return state for some $m > 1$.

State i is said to be aperiodic if $d_i = 1$

State i is said to be periodic if $d_i > 1$.

A state i is said to be persistent or recurrent if the return to state i is certain.

The state i is said to be transient if the return to state i is uncertain.

If the state i is finite, then it is said to be non null persistent and null persistent if state i is infinite.

A non null persistent and aperiodic state is called ergodic.

If a Markov chain is finite irreducible, all its states are non null persistent.

A state is said to be an absorbing state if no other state is accessible from it.

Problems.

1) The transition probability matrix of a Markov chain $\{X_n\}$, $n=1,2,3,----$ having 3 states 1, 2 and 3 is

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

and the initial distribution is $p^{(0)} = (0.7 \quad 0.2 \quad 0.1)$

Find i) $P[X_2=3]$ ii) $P[X_3=2, X_2=3, X_1=3, X_0=2]$

Sol. i) $p^{(0)} = (0.7 \quad 0.2 \quad 0.1)$

$$p^{(1)} = p^{(0)}P = (0.7 \quad 0.2 \quad 0.1)\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= (0.22 \quad 0.43 \quad 0.35)$$

$$p^{(2)} = p^{(1)}P = (0.22 \ 0.43 \ 0.35)\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= (0.385 \quad 0.336 \quad 0.279)$$

$$\therefore P(X_2=3) = 0.279$$

$$\boxed{\begin{array}{l} P(A/B) = \dfrac{P(A\cap B)}{P(B)} \\ P(A\cap B) = P(A/B)\cdot P(B) \end{array}}$$

ii) $P[X_3=2, X_2=3, X_1=3, X_0=2]$

$$= P[X_3=2/X_2=3, X_1=3, X_0=2] \cdot P[X_2=3, X_1=3, X_0=2]$$

$$= P[X_3=2/X_2=3] \cdot P[X_2=3/X_1=3, X_0=2] \cdot P[X_1=3, X_0=2]$$

$$= P_{32}^{1} \cdot P[X_2=3/X_1=3] \cdot P[X_1=3/X_0=2] \cdot P(X_0=2)$$

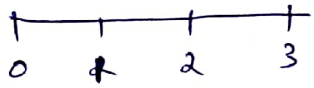
$$= P_{32}^{1} \, P_{33}^{1} \cdot P_{23}^{1} \cdot P(X_0=2)$$

$$= (0.4)(0.3)(0.3)(0.2) = \underline{0.0048}$$

2) A man is at an integral point on the x-axis between the origin and the point 3. He takes a unit step to the right with probability $\frac{1}{3}$ or to the left with probability $\frac{2}{3}$, unless he is at the origin, where he takes a step to the right to reach the point 1 or is at the point 3, where he takes a step to the left to reach the point 2. What is the probability that i) he is at the point 1 after 3 walks?

ii) he is at the point 1 in the long run?

Sol: The transition probability matrix is



$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 2 & 0 & 2/3 & 0 & 1/3 \\ 3 & 0 & 0 & 1 & 0 \end{array}$$

Initial prob. distribution is

$$p^{(0)} = \left(\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \right)$$

i) $P(\text{point 1 after 3 walks}) = P(\underset{\text{Stage or steps.}}{X_3} = \overset{\text{state}}{1}) = ?$

$$p^{(1)} = p^{(0)} P = \left(\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \right) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \left(\frac{1}{6} \quad \frac{5}{12} \quad \frac{1}{3} \quad \frac{1}{12} \right)$$

$$p^{(2)} = p^{(1)}p = \left(\frac{1}{6} \quad \frac{5}{12} \quad \frac{1}{3} \quad \frac{1}{12}\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \left(\frac{5}{18} \quad \frac{7}{18} \quad \frac{2}{9} \quad \frac{1}{9}\right)$$

$$p^{(3)} = p^{(2)}p = \left(\frac{5}{18} \quad \frac{7}{18} \quad \frac{2}{9} \quad \frac{1}{9}\right)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \left(\frac{7}{27} \quad \frac{23}{54} \quad \frac{13}{54} \quad \frac{2}{27}\right)$$

$$\therefore p(X_3 = 1) = \frac{23}{54}$$

ii) Long run - limiting case

Let $\pi = (\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3)$ be the steady-state probability. Using the property of π, we have

$$\pi p = \pi$$

$$(\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3)$$

$$\Rightarrow \frac{2}{3}\pi_1 = \pi_0 \text{ ——— } ①$$

$$\pi_0 + \frac{2}{3}\pi_2 = \pi_1 \text{ ——— } ②$$

$$\frac{1}{3}\pi_1 + \pi_3 = \pi_2 \text{ ——— } ③$$

$$\frac{1}{3}\pi_2 = \pi_3 \text{ ——— } ④$$

Also $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ ——— ⑤

① $\Rightarrow$ $\pi_1 = \frac{3}{2}\pi_0$

② $\Rightarrow$ $\frac{2}{3}\pi_2 = \pi_1 - \pi_0$

$= \frac{3}{2}\pi_0 - \pi_0$

$= \frac{1}{2}\pi_0$

$\pi_2 = \frac{3}{4}\pi_0$

$\pi_3 = \frac{1}{3}\pi_2 = \frac{1}{3}\cdot\frac{3}{4}\pi_0 = \frac{1}{4}\pi_0$

⑤ $\Rightarrow$ $\pi_0 + \frac{3}{2}\pi_0 + \frac{3}{4}\pi_0 + \frac{1}{4}\pi_0 = 1$

$\frac{14}{4}\pi_0 = 1$

$\pi_0 = \frac{2}{7}$.

$\pi_1 = \frac{3}{2}\pi_0 = \frac{3}{7}$

$\pi_2 = \frac{3}{4}\pi_0 = \frac{\cancel{2}3}{14}$, $\pi_3 = \frac{1}{14}$

$\therefore \pi = \left(\frac{2}{7} \quad \frac{3}{7} \quad \frac{3}{14} \quad \frac{1}{14}\right)$

$\therefore$ P (Point 1 in the long run) $= \frac{3}{7}$

---

3) A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die and drove to work if and only if a 6 appeared. Find i) the prob. that he takes a train on 3rd day and ii) the prob. that he drives to work in the long run.

Sol. TPM of the chain is $P = \begin{array}{c|cc} & T & C \\ \hline T & 0 & 1 \\ C & 1/2 & 1/2 \end{array}$

The initial state prob. dist. is $p^{(1)} = \left(\frac{5}{6} \quad \frac{1}{6}\right)$

$$p^{(2)} = p^{(1)}P = \left(5/6 \quad 1/6\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{1}{12} \quad \frac{11}{12}\right)$$

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12} \quad \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{11}{24} \quad \frac{13}{24}\right)$$

$\therefore$ P(the man travels by train on 3rd day) $= \frac{11}{24}$

ii) Let $\pi = (\pi_1 \quad \pi_2)$ be the steady state probability.

By the property of π,

$$\pi P = \pi$$

$$(\pi_1 \quad \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1 \quad \pi_2)$$

$\Rightarrow \frac{1}{2}\pi_2 = \pi_1$ ——— ①

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$ ——— ②

Also $\pi_1 + \pi_2 = 1$ ——— ③

① $\Rightarrow \pi_2 = 2\pi_1$

③ $\Rightarrow \pi_1 + 2\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{3}$

$\therefore \pi_2 = \frac{2}{3}$

$\therefore \pi = \left(\frac{1}{3} \quad \frac{2}{3}\right)$

$\therefore$ P(the man travels by car in the long run) $= \frac{2}{3}$.

4) A housewife buys 3 kinds of cereals A, B and C. She never buys the same cereal in successive weeks. If she buys cereal A, the next week she buys cereal B. However if she buys B & C, the next week she is 3 times as likely to buy A as the other cereal. In the long run, how often she buys each of the three cereals?

Sol. Tpm is
$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 3/4 & 0 & 1/4 \\ C & 3/4 & 1/4 & 0 \end{array}$$

Let $\pi = (\pi_1 \quad \pi_2 \quad \pi_3)$ be the steady state prob.

By the property of π,

$$\pi P = \pi$$

$$(\pi_1 \quad \pi_2 \quad \pi_3)\begin{pmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{pmatrix} = (\pi_1 \quad \pi_2 \quad \pi_3)$$

$$\Rightarrow \frac{3}{4}\pi_2 + \frac{3}{4}\pi_3 = \pi_1 \quad \text{——— ①}$$

$$\pi_1 + \frac{1}{4}\pi_3 = \pi_2 \quad \text{——— ②}$$

$$\frac{1}{4}\pi_2 = \pi_3 \quad \text{——— ③}$$

Also $\pi_1 + \pi_2 + \pi_3 = 1$ ——— ④

$$② \Rightarrow \pi_1 = \pi_2 - \frac{1}{4}\pi_3 = \pi_2 - \frac{1}{4}\left(\frac{1}{4}\pi_2\right) = \frac{15}{16}\pi_2$$

$$④ \Rightarrow \frac{15}{16}\pi_2 + \pi_2 + \frac{1}{4}\pi_2 = 1 \Rightarrow \pi_2 = \frac{16}{35}$$

$$\pi_3 = \frac{4}{35}, \quad \pi_1 = \frac{15}{35}$$

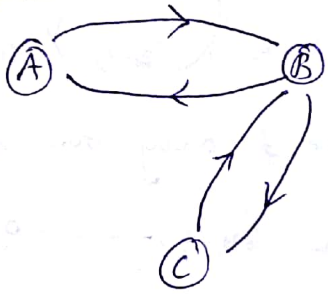
$\therefore \pi = \left(\frac{15}{35} \quad \frac{16}{35} \quad \frac{4}{35}\right)$ is long run probability.

5) Find the nature of the states of the Markov chain with the tpm $p = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$

Sol.

$$P = \begin{array}{c|ccc} & A & B & C \\ A & 0 & 1 & 0 \\ B & 1/2 & 0 & 1/2 \\ C & 0 & 1 & 0 \end{array}$$

Markov diagram



The chain is irreducible.

Only 3 states. So finite and irreducible

∴ All its states are non null persistent

Period of A = GCD$\{2,4,6,---\}$ = 2

∴ State A is not ergodic

Period of B = GCD$\{2,4,6,---\}$ = 2

∴ State B is not ergodic

Period of C = GCD$\{2,4,6,---\}$ = 2

∴ State C is not ergodic.

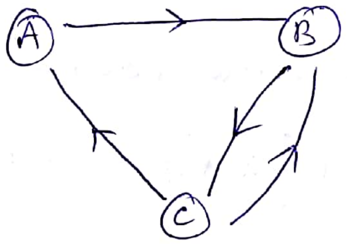
6) Three boys A, B and C are throwing a ball to each other. A always throw the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states.

Sol. TPM is
$$P = \begin{array}{c|ccc} & A & B & C \\ A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

States of X_n depend only on states of X_{n-1}, but not on states of $X_{n-2}, X_{n-3}, \ldots$ or earlier states.

$\therefore \{X_n\}$ is a Markov chain.

Markov diagram



The chain is irreducible and it is finite.

$\therefore$ All its states are non-null persistent.

Period of $A = GCD\{3,5,6,7,\ldots\} = 1$

$\therefore$ State A is aperiodic

Period of $B = GCD\{2,3,4,\ldots\} = 1$

State B is aperiodic

Period of $C = GCD\{2,3,4,\ldots\} = 1$

State C is aperiodic.

$\therefore$ All its states are ergodic.

7) A fair coin is tossed until 3 heads occur in a row. Let X_n be the sequence of heads ending at the n^{th} trial. What is the probability that there are atleast 8 tosses of the coin?

Sol. Tpm is $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Probability that there are atleast 8 tosses of the coin $\Big\} = P(X_7 = 0 \text{ or } 1 \text{ or } 2)$

The initial prob. dist. is $p^{(0)} = \begin{pmatrix} \overset{0}{1} & \overset{1}{0} & \overset{2}{0} & \overset{3}{0} \end{pmatrix}$

$$p^{(1)} = p^{(0)} P = (1 \quad 0 \quad 0 \quad 0) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\tfrac{1}{2} \quad \tfrac{1}{2} \quad 0 \quad 0\right)$$

$$p^{(2)} = p^{(1)} P = \left(\tfrac{1}{2} \quad \tfrac{1}{2} \quad 0 \quad 0\right) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad 0\right)$$

$$p^{(3)} = p^{(2)} P = \left(\tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad 0\right) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{1}{8}\right)$$

$$p^{(4)} = p^{(3)} P = \left(\tfrac{1}{2} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{1}{8}\right) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\tfrac{7}{16} \quad \tfrac{1}{4} \quad \tfrac{1}{8} \quad \tfrac{3}{16}\right)$$

$$p^{(5)} = p^{(4)} p = \left(\frac{7}{16} \ \frac{1}{4} \ \frac{1}{8} \ \frac{3}{16}\right) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\frac{13}{32} \ \frac{7}{32} \ \frac{1}{8} \ \frac{1}{4}\right)$$

$$p^{(6)} = p^{(5)} p = \left(\frac{13}{32} \ \frac{7}{32} \ \frac{1}{8} \ \frac{1}{4}\right) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\frac{3}{8} \ \frac{13}{64} \ \frac{7}{64} \ \frac{5}{16}\right)$$

$$p^{(7)} = p^{(6)} p = \left(\frac{3}{8} \ \frac{13}{64} \ \frac{7}{64} \ \frac{5}{16}\right) \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left(\frac{11}{32} \ \frac{3}{16} \ \frac{13}{128} \ \frac{47}{128}\right)$$

$$P(X_7 = 0 \text{ or } 1 \text{ or } 2) = \frac{11}{32} + \frac{3}{16} + \frac{13}{128}$$

$$= \frac{44+24+13}{128}$$

$$= \frac{81}{128}$$